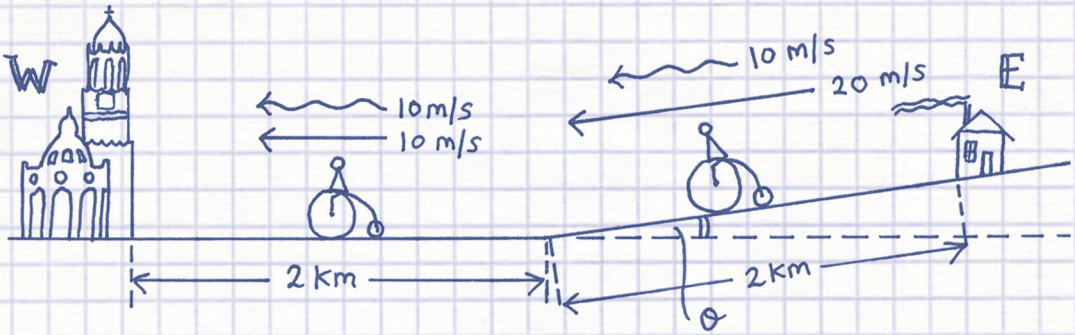
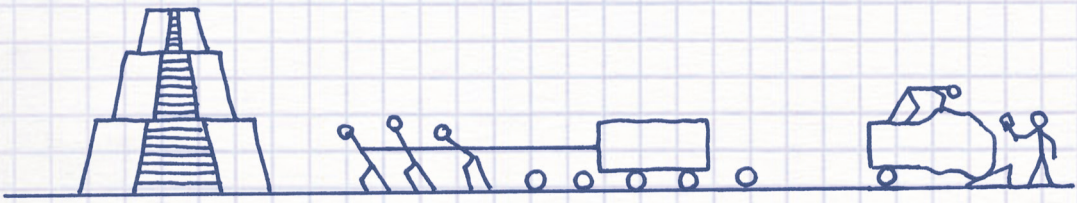
THOMAS POVEY



Professor Povey's Perplexing Problems



Pre-University Physics and Maths Puzzles with Solutions

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Professor Povey's Perplexing Problems

Pre-University Physics and Maths Puzzles with Solutions

Thomas Povey

ONEWORLD

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To Mike Leask and Terry Jones—brilliant teachers, insatiable adventurers.
With thanks to Tet Amaya, for substantial contributions throughout.

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Introduction

This is a personal book which celebrates a passion I have always had for playful problems in physics and maths. Questions that rely on material no harder than that studied in high school, but which encourage original thinking, and stretch us in new and interesting directions.

In practically every country that has an established higher education system, the most competitive universities operate some kind of admissions test or interview. These generally have only one object: to distinguish from many brilliant applicants those with the greatest potential. To test this potential, questions are often designed to test an applicant's ability to think creatively. In many countries, these pre-university tests are set by university professors. Whilst the questions might reference a particular high-school syllabus, they would not normally be constrained by it. The questions are often deliberately off-beat. This is to see if applicants can apply what they have already learnt in new and more challenging situations. The questions test the applicants' ability to pick an unusual problem apart, reduce it to its essential components, and use standard tools to solve it.

This book is a collection of some of my favourite pre-university problems in physics and maths. These questions are devised to encourage curiosity and playfulness, and many are of the standard expected in some university entrance tests. These questions are perplexing, puzzling, but—most of all—fun. You should regard them like *toys*. Pick up the one that most appeals, and play with it. When you have exhausted it, you can entertain a friend with it. It might seem impossibly nerdy, but for me almost nothing is as enjoyable as being baffled by an apparently simple problem in an area of classical physics I *thought* I understood. This book is a way of sharing the pleasure I have taken in some of the physics and maths puzzles I have most enjoyed. The questions will appeal to those high-school students who have mastered the basics, and feel they have room to play a little with something more unusual. Teachers can also use the problems to stretch their pupils with something more challenging or unconventional. Here I am grateful to those teachers who allowed their classes to pilot these problems, and to the students who bravely attempted them.

I invented many of the questions myself. Others were suggested or inspired by friends and colleagues with an interest in this book. Still more are

well-known classics. Even working outside a syllabus however, it is hard to be entirely original with questions in physics and maths. Although these questions can be asked in essentially unlimited ways, the number of fundamental concepts underpinning them is relatively small. Even the ones I am most proud of will no doubt have been asked in numerous other forms over the years.

There was one problem I was unable to solve. Many of these questions—especially the harder ones—work best as *discussion problems*. They are not meant to be asked and answered straight. Instead they should be discussed as part of a tutorial-style conversation, with an experienced tutor helping the student along, giving hints, correcting mistakes, providing encouragement, and so on. There is no real substitute for this kind of help. A question which might seem impenetrable can suddenly yield when a tutor provides just the right hint. I would encourage you to try the problems with friends, or subject tutors, or anyone with a sound knowledge of physics and maths.

I should say a few words about the level of difficulty of the problems. Every single one of them is challenging in some way. The publisher asked me to use stars to indicate their relative difficulty. Although I think this is quite subjective, I had to concede that it would be a shame for someone to happen to try all the most difficult problems first and feel defeated. So I have indicated how hard *I* found the problems the first time I tried them. If you think I have rated some of them too generously, it simply means your intuition is better than mine in that particular area. In general, those questions with one or two stars should yield to a student who is on top of the pre-university high-school syllabus[1] in physics and maths. The following descriptive gradings should give you some idea of what to expect.

> ★ Not too difficult. Requires thought, and some insight, but should be soluble without hints.
>
> ★★ Challenging. Requires considerable thought, and some insight, and may require simple hints.
>
> ★★★ Hard. Requires significant thought, and considerable insight, and more substantial hints. In some cases, questions with three stars are only suitable as discussion problems.

I have also included a handful of exceptionally hard questions, to give the most confident students an opportunity for a real challenge.

> ★★★★ Exceptionally hard. Suitable for discussion. May be too difficult for most students to solve without substantial help. Complex and unusual, these questions allow for different and non-standard approaches.

[1]Here I mean a student who has completed the first year of A-level, International Baccalaureate, Scottish Highers, SAT, etc., and is about a year away from entering higher education.

This book has been what I call my "Saturday project", and there are, I increasingly discover, always fewer Saturdays than I hope there will be. I have not had the luxury of someone to work with, so there will certainly be mistakes, and errors of logic. They are all my fault. But I hope they are few enough that this book brings more enlightenment than confusion. The most challenging aspect of putting it together was striking the right balance with the difficulty of questions. The questions range from being quite straightforward to really quite hard—they are as unpredictable as many university entrance tests. Here I am indebted to the many teachers, students and colleagues who were kind enough to provide detailed comments on the difficulty of the questions. I would be delighted to receive further constructive notes and any genuinely unusual questions which could improve a future edition of this book, and thank in advance anyone willing to provide this feedback.[2] I hope you will find friends you can share the fun of these questions with, and argue the case for improved or more elegant solutions (many of which exist). To share your solutions with a wider audience, and to learn from the ideas of others, you can contribute to the Puzzle Forum at

www.PerplexingProblems.com

If you are feeling brave, you can also play The Deadly Game of Puzzle Points, which is explained in the final chapter of this book. I've provided scoresheets and ticklists so you can keep track of your progress.

Finally, good luck, and have fun.

T.P., Oxford, 2015

[2] You can contact me by emailing tom@tompovey.com

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I am indebted to all the teachers, headteachers, students, parents of students, and colleagues who provided many useful comments, particularly on the difficulty and range of questions that were eventually included in this book. Among those who were kind enough to provide feedback were: Oscar Van Nooijen, Brian Smith, Terry Jones, Frances Burge, Julie Summers, Robin Parmiter, Elaine Cook, Georgina Allan and Glenn Black. With thanks to Ben Sumner for copyediting the manuscript.

Chapter 1

Geometry

In this chapter we look at problems in *geometry*.[1] This ancient branch of mathematics dates back to approximately 600 BC, but advanced most rapidly around the time of Euclid,[2] who developed results into a logical system based on a set of simple axioms.[3] Well over two thousand years later, what is now known as *Euclidean geometry*[4] is still taught in schools. It is interesting to think that when we solve problems in this area we often do so with exactly the same logic, tools and methods as the ancient Greeks used. So we are concerned with developing simple but clever methods for geometric calculations (areas and volumes, for example) without recourse to integration. That is, without the need to represent the geometry with explicit functions which are dealt with using calculus. The latter was developed two thousand years after Euclid—in the seventeenth century—by Isaac Newton and Gottfried Leibniz.

Some of these puzzles are well-known, others are based on well-known questions, and still more I invented in the last few years. My personal favourite is the chess board and hoop question, which I came up with three years ago, and which students have found quite satisfying and enjoyable to solve.

[1] Literally translated from the ancient Greek as *earth measurement*.

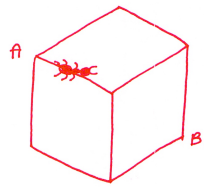
[2] Euclid of Alexandria (c. 300 BC) was a Greek mathematician most famous for Elements, a collection of works on geometry, number theory and logic, said by some to be the most influential mathematical work ever written. Euclid is known as the Father of Geometry.

[3] An *axiom* is a premise that is assumed to be so simple and obvious that it is self-evidently correct. Euclid's five axioms were as follows: i) A straight line can be drawn from any point to any other point; ii) a line segment can be extended continuously in both directions; iii) a circle can be drawn with any centre and any radius; iv) all right angles are equal; v) if a straight line falls on two straight lines and makes interior angles on the same side that sum to less than the sum of two right angles, the two straight lines, if extended indefinitely, intersect on that side (this is known as the *parallel postulate*).

[4] *Non-Euclidean* geometries, such as hyperbolic and elliptic geometries, were developed in the nineteenth century by relaxing the parallel postulate.

1.1 Shortest walk ★

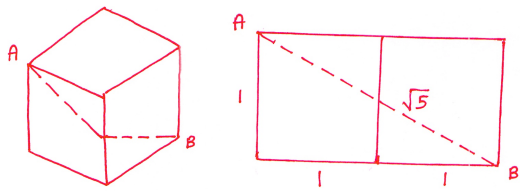
AN ANT STARTS at one vertex of a solid cube with side of unity length. Calculate the distance of the shortest route the ant can take to the furthest vertex from the starting point. There is more than one solution to this problem.



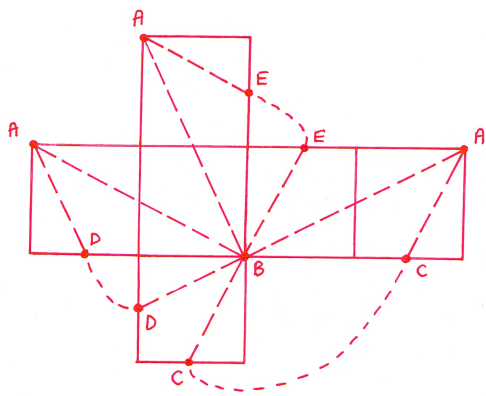
Answer

This is a well-known question, though not one with very much scope for discussion.

If we consider unwrapping any pair of sides that connect the starting vertex A to the furthest vertex from the starting vertex, B, it is clear that the unwrapped surface forms a rectangle with sides of length 1 and 2. The shortest distance is a straight line with length $\sqrt{5}$.



This solution is not unique. There are, in fact, six solutions of the same minimum length, which are shown in this diagram of the unwrapped cube.



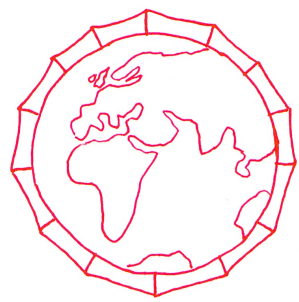
1.2 Intercontinental telephone cable ★

This rather neat little question is simple enough that it does not warrant much introduction. I have heard of a number of variants over the years.

A TELEPHONE COMPANY has run a very long telephone cable all the way round the middle of the Earth. Assuming the Earth to be a sphere, and without recourse to pen and paper, estimate how much additional cable would be required to raise the telephone cable to the top of 10 m tall telephone poles.

Answer

The circumference of the Earth is given by $C = 2\pi R_E$, where R_E is the radius of the Earth. The new circumference at the top of 10 m tall telephone poles is given by $C' = 2\pi\,(R_E + 10)$. The difference in the circumferences is $C' - C = 20\pi$. We should know that $\pi \approx 3.14$, so we should be able to work out in our heads that the additional length of cable is approximately equal to 62.8 m. The question is designed to be slightly tricky in that we might be expecting a big number. If we approach it rationally we see that questions don't come much easier than this!



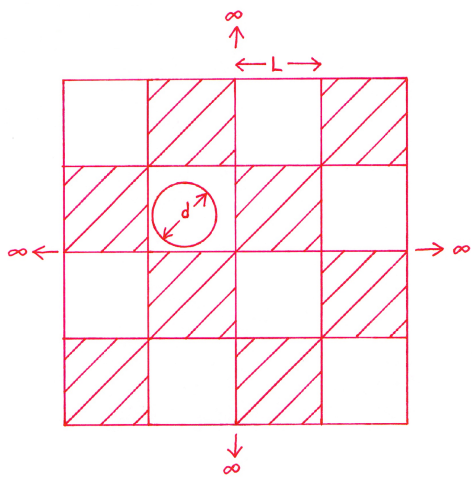
There are, of course, intercontinental telephone cables, and the history of submarine cable-laying started not long after the invention of the telegraph. The first commercial telegraph was developed by William Cooke and Charles Wheatstone in 1838, and as early as 1840 Samuel Morse was working on the concept of submerged cables for transatlantic operation. Experiments under

the waters of New York Harbour in 1842 confirmed the feasibility of such a project, and in 1850 the first telephone cable was laid under the English Channel (to France), by the Anglo-French Telegraph Company. Numerous other submerged telephone cables were laid across the English Channel, the Irish Sea and the North Sea in the years that followed, and the first attempt at a transatlantic cable was made in 1858. The first successful transatlantic project was in 1865, when undersea cables were laid by the SS Great Eastern, a huge iron steamship designed by the Bristolian Isambard Kingdom Brunel. Now there are numerous fibre-optic cables snaking across the Atlantic seabed.

1.3 Chessboard and hoop ★★

I invented this puzzle a couple of years ago. It is one of my favourites because students seem to enjoy it, and can generally make good progress without much help, although sometimes they need a hint to get started.

A THIN HOOP of diameter d is thrown on to an infinitely large chessboard with squares of side L. What is the chance of the hoop enclosing two colours?



Answer

This question can be answered by applying very simple geometry, and should be accessible to anyone with a good grasp of GCSE-level maths. The first realisation is that if the hoop crosses any tile boundary it will enclose two colours. It is easier to first pose the reverse of the question asked—that is, "what is the chance of not enclosing two colours?" We then subtract this chance from the chance of *any* outcome, giving the answer to the question as posed. We will use notation familiar to students of *probability*, although

only the most elementary understanding of that subject is required here. We use the notation $P(A)$ to denote the probability of the hoop enclosing two colours. By definition $0 \leq P(A) \leq 1$; that is, the probability lies somewhere between 0 (will not happen) and 1 (will definitely happen). We use $P(\bar{A})$ to denote the probability of the hoop *not* enclosing two colours. There are only two possible outcomes, so $P(\bar{A}) = 1 - P(A)$.

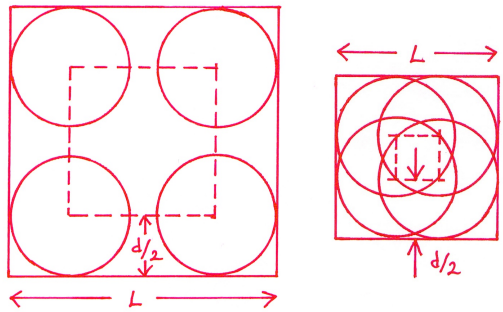
We are dealing with two-dimensional space, so we will probably need to define the landing *areas* which lead to one outcome or the other. If we were dealing with three-dimensional space we would need to define *volumes*. In one-dimensional space, we would need to define *distances*, and so on. We can define the location of the hoop unambiguously by referring to the position of the centre of the hoop. We could equally use the location of any identifiable point on the rim of the hoop, but then we would also need to express the angular orientation of the hoop. This introduces more complexity with no benefit.

We will now consider three cases: $d > L$; the special case $d = L$; and $d < L$. Let's consider each in turn.

- THE CASE OF $d > L$. In this case, in which the diameter is larger than the side of a tile, it is perfectly obvious that the hoop must enclose two colours. We see that $P(A) = 1$.

- THE SPECIAL CASE OF $d = L$. For the special case that $d = L$, it is possible for the hoop to land entirely within a single tile, but the chance of this happening is $1/\infty$. In this *limiting case* the circle touches the tile at four points. That is, a single solution out of infinitely many possible solutions satisfies the condition that only one colour is enclosed. We write $P(\bar{A}) = 1$, and therefore $P(A) = 0$.

- THE CASE OF $d < L$. Consider all the circles that can be drawn entirely within a single tile, and which touch the tile boundary either at one point (along an edge of the tile) or at two points (in the corner of the tile). For now we observe that the circles that touch the edges of the tile at either one or two points have centres which lie on a smaller square exactly in the middle of the larger tile. The *locus* of points is shown in the diagram. For a circle of diameter d, the shortest distance between the locus of points and the tile is $d/2$ everywhere. The side of the smaller square is therefore $L - d$.



We now observe that any circle with a centre within the smaller square does not touch the tile boundaries. The locus of all points at which the hoop centre can be without touching the border of the tile is defined by the smaller square. The probability of the hoop centre landing in the smaller square is given by the ratio of the area of the smaller square to that of the larger square.

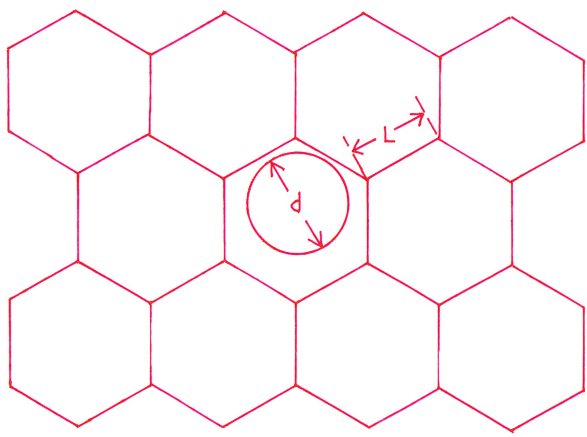
$$P(\bar{A}) = \frac{(L-d)^2}{L^2} \text{ for } d \leq L$$

So

$$P(A) = 1 - \frac{(L-d)^2}{L^2} \text{ for } d \leq L$$

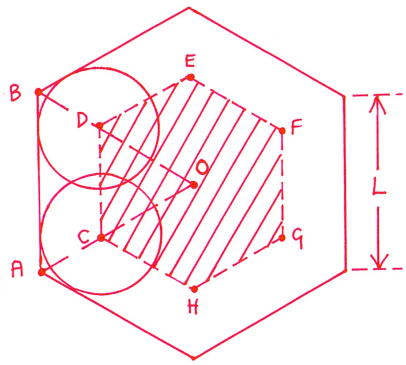
1.4 Hexagonal tiles and hoop ★★

An infinitely large floor is tiled with regular hexagonal tiles of side L. Different colours of tiles are used so that no two tiles of the same colour touch. A hoop of diameter d is thrown onto the tiles. What is the chance of the hoop enclosing more than one colour?

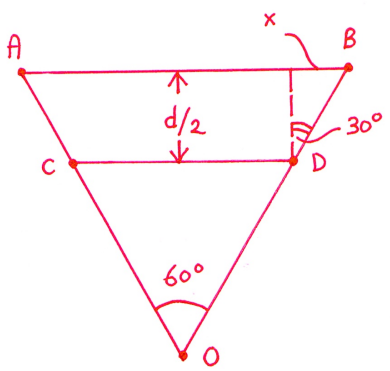


Answer

We can answer this question using a similar method to that used for the chessboard and hoop, but the geometry is a little more complex. We start by defining the locus of hoop centres that allows a hoop entirely enclosed within a tile to touch the edge of the tile at one position (edges) or two positions (corners). The locus of such points is defined by the line $CDEFGH$, which describes a smaller hexagon centred on the tile centre. The edges of the smaller hexagon are $d/2$ away from the edges of the large hexagon. That is, the perpendicular distance between AB and CD, for example, is $d/2$. The locus of all hoop centres for which the hoop does not intersect the edge of the tile is defined by the *area* $CDEFGH$.



To calculate the area of the smaller hexagon we need the length of the side CD, which is denoted by $|CD|$. From the geometry of the regular hexagons we can see that $|CD| = |AB| - 2x = L - 2x$, where $x = (d/2)\tan 30° = d\left(\sqrt{3}/6\right)$. We have $|CD| = L - d\left(\sqrt{3}/3\right)$.



The chance of landing entirely within a tile, $P(\bar{A})$, is given by the ratio of the areas of the smaller tile and the larger tile. We do not need to calculate

the areas explicitly, because we know they are of similar shape, so the area is proportional to the square of any *characteristic length*. For the characteristic length we could take the side length. The relevant lengths become $|CD|$ and $|AB|$. This gives

$$P(\bar{A}) = \frac{|CD|^2}{|AB|^2} = \frac{\left(L - \frac{d\sqrt{3}}{3}\right)^2}{L^2} \text{ for } d \leq \sqrt{3}L$$

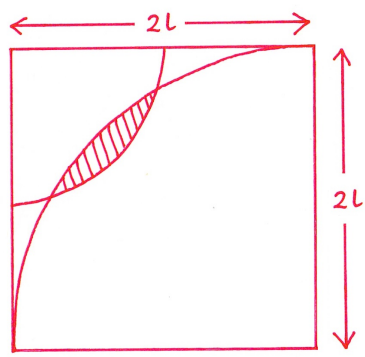
The probability of enclosing more than one colour, $P(A)$, is given as $P(A) = 1 - P(\bar{A})$, or

$$P(A) = 1 - \frac{\left(L - \frac{d\sqrt{3}}{3}\right)^2}{L^2} \text{ for } d \leq \sqrt{3}L$$

1.5 Intersecting circles ★★

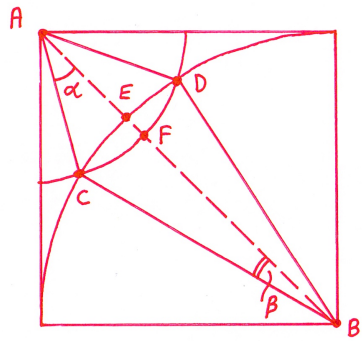
A friend of mine introduced me to this puzzle some years ago. I like it because it is deceptively simple-looking, but it actually took me some time to solve it. I think I simply started down the wrong track, and took some time to find my way back to a sensible method. Hopefully you will find it easier than I did.

Two circles of radius l and $2l$ intersect as shown. What is the area of the shaded region?

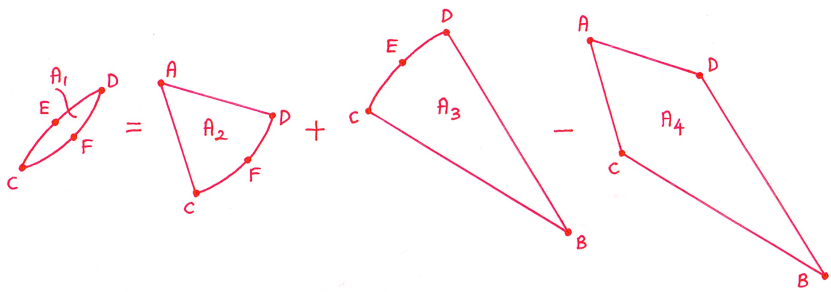


Answer

Consider first the construction of points A, B, C, D, E and F in the sketch below. We note first that the distance between A and C is equal to l, because AC forms a radius of the smaller circle. We write $|AC| = |AD| = l$. A similar statement leads to $|BC| = |BD| = 2l$.



The area of the shaded element $CEDFC$ is equal to the area of sector $ADFCA$ plus the area of sector $BCEDB$ minus the area of the quadrilateral $BCADB$.



We represent these areas by A_1 to A_4 respectively, and write

$$A_1 = A_2 + A_3 - A_4$$

Expressing this explicitly we have

$$A_1 = l^2\alpha + 4l^2\beta - |AB|\,\frac{|CD|}{2}$$

To simplify this we require expressions for $|AB|$, $|CD|/2$, α and β. We now find each of these in turn.

- FIND $|AB|$. We see that AB is the longest diagonal of the square, and has length $|AB| = 2\sqrt{2}l$.
- FIND α. Using the cosine rule[5] on the triangle $ADBA$ we see that

$$(2l)^2 = l^2 + \left(2\sqrt{2}l\right)^2 - 2\,(l)\left(2\sqrt{2}l\right)\cos\alpha$$

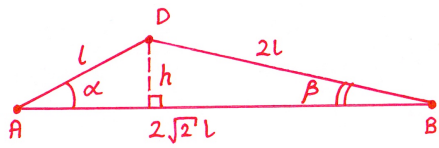
Simplifying, we get

[5] Usually written in the form $c^2 = a^2 + b^2 - 2ab\cos C$, where a, b and c are the lengths of the sides of a triangle, and C is the angle between the sides of length a and b.

$$4 = 1 + 8 - 4\sqrt{2}\cos\alpha$$

That is

$$\cos\alpha = \frac{5\sqrt{2}}{8}$$



- FIND β. We find β using an identical method to the one we used to find α, and get

$$\cos\beta = \frac{11\sqrt{2}}{16}$$

- FIND $|CD|/2$. We set $|CD|/2 = h$, and note h in the construction $ADBA$. From the construction we see that

$$h = l\sin\alpha = l\sqrt{1 - \cos^2(\alpha)} = \frac{l\sqrt{14}}{8}$$

Now that we have expressions for $|AB|$, $|CD|/2$, α and β, we can evaluate A_1. We write

$$A_1 = l^2\left[\cos^{-1}\left(\frac{5\sqrt{2}}{8}\right) + 4\cos^{-1}\left(\frac{11\sqrt{2}}{16}\right) - \frac{\sqrt{7}}{2}\right]$$

This gives the area of the shaded region as

$$A_1 \approx 0.108\, l^2$$

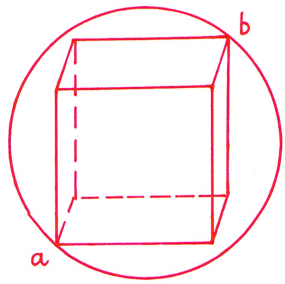
My original method was considerably longer even than this! I like this problem because it is apparently trivial, but it takes some thought to solve with relatively compact algebra.

1.6 Cube within sphere ★

WHAT IS THE volume of the largest cube that fits entirely within a sphere of unity volume?

Answer

By symmetry, all eight vertices of the cube must touch the sphere. Again, by symmetry, the length of the longest diagonals of the cube, for example $|ab|$, must be equal to the diameter of the sphere D, because the line ab passes through the centre of the sphere.

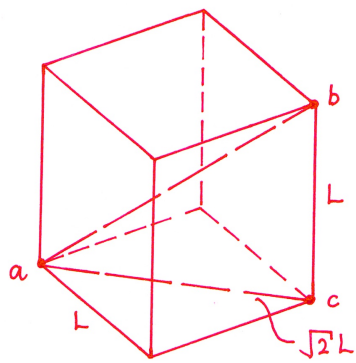


From the geometry of the cube, using Pythagoras' theorem

$$|ab|^2 = \left(\sqrt{2}L\right)^2 + L^2$$

where L is the length of the side of the cube. Thus

$$|ab| = \sqrt{3}L = D$$



The volume of the cube is thus given by

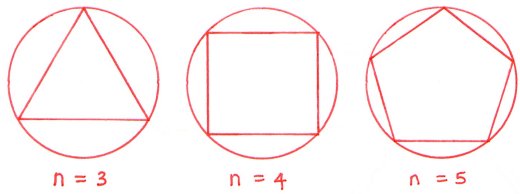
$$V_C = L^3 = \frac{D^3}{3^{3/2}}$$

All that remains is to substitute for D^3 using the equation for the volume of the sphere, $V_S = (4/3)\,\pi r^3 = \pi D^3/6 = 1$. Rearranging $D^3 = 6/\pi$. The volume of the cube within the sphere is given by

$$V_C = \frac{2}{\pi\sqrt{3}} \approx 0.368$$

1.7 Polygon inscribed within circle ★

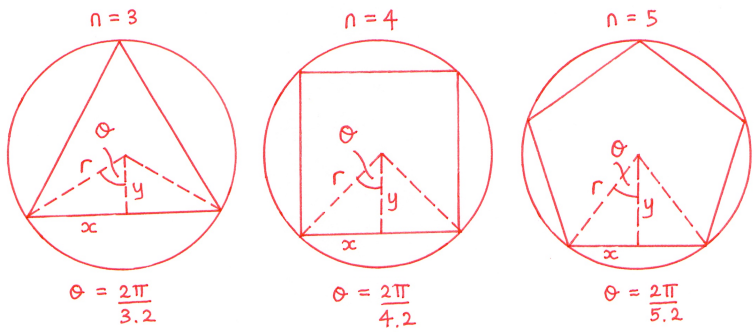
WHAT IS THE area of an n-sided regular polygon inscribed within a circle of radius r?



Answer

This is a straightforward problem really, but one that is a good test of a student's ability to derive simple formulae for unfamiliar situations. There is relatively little scope for practising this skill in most high-school courses, so I've included a number of problems of this type to develop this aspect of problem-solving.

Consider the constructions for $n = 3, 4$ and 5-sided polygons.



The polygons are regular, so using their symmetry we can decompose them into $2n$ right-angled triangles, each with area $(xy)/2$. The total area of the n-sided polygon becomes

$$A_n = 2n\frac{xy}{2} = nxy$$

Noting the angle θ in each of the constructions, we see that $\sin\theta = x/r$ and $\cos\theta = y/r$, where $\theta = (2\pi)/(2n) = \pi/n$. The area of the polygon now becomes

$$A_n = nr^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n}$$

We can simplify this using the trigonometric identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$. We get

$$A_n = \frac{nr^2}{2} \sin \frac{2\pi}{n}$$

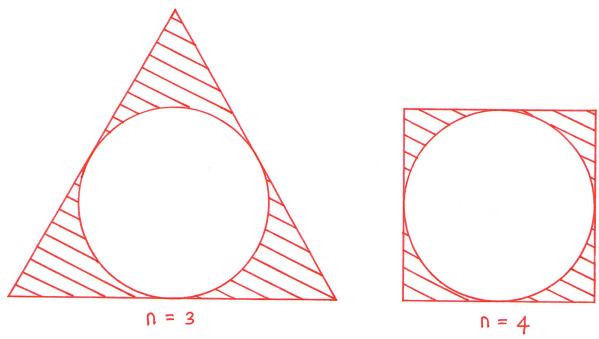
As a final note, we see that, for very large n, we can make the approximation $\sin(2\pi/n) \approx 2\pi/n$. The expression for area becomes

$$A_n \approx \pi r^2 \text{ (for large } n)$$

This is the equation for the area of a circle, as expected.

1.8 Circle inscribed within polygon ★★★

FOR A CIRCLE inscribed inside a regular n-sided polygon, what is the minimum n so that $A_{shaded}/A_{circle} \leq 1/1000$?



This is a variant on the previous question, but has the added problem that the final expression cannot be solved by simply rearranging the equation. You can use a number of techniques familiar to high-school students, however. And, unusually, the suggested methods are now given as hints.

One method is to use a graphical solution—that is, to find two expressions so that the intersection defines the solution. A second method is an iterative numerical technique—essentially trial and error of a thoughtful sort, in which you see if you are getting closer or further from the solution and correct your guess up or down accordingly until you *converge* on a solution. A third technique, and certainly the most elegant, is to use a *Taylor polynomial approximation* to the function. *Taylor series* are no longer taught in all high-school syllabuses but are certainly taught in some extension papers, and are covered in the first year of all university courses. Because the Taylor approximation is such an elegant technique we will say a few words about it here.

The first thing to say is that the concept of a Taylor series is very simple, so there is no need to be intimidated by it. It says we can represent a function about a particular point as an infinite polynomial series. If, for example,

we wish to represent $\sin x$ around zero, we start with a linear (or *first-order*) approximation that only applies at very small angles, namely

$$\sin x \approx x \quad \text{for very small } x$$

This is an approximation that all high-school students should be familiar with,[6] and will use routinely. As we move away from zero the approximation starts to fail because, as we well know, the graph of $y = \sin x$ has curvature as we move away from the origin. To correct for the curvature, we need to add a higher-order term which itself is approximately zero when very close to the origin, so as not to harm our first-order approximation, but which gets larger (either positive or negative in sign) at a greater rate than x as we move away from the origin. We skip the second-order term because in this series it is zero, so the third-order approximation[7] is

$$\sin x \approx x - \frac{x^3}{3!}$$

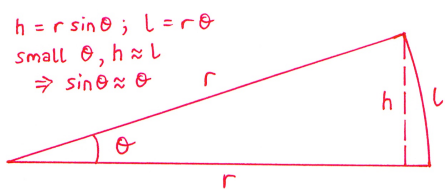
where 3! stands for *three factorial*, or $3 \times 2 \times 1 = 6$. The factorial[8] of any integer n is simply the product of all positive integers less than or equal to n. The third-order approximation extends the region around the origin in which the approximation is accurate. We can keep adding terms to get what is called the Taylor series for $\sin x$, which is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

and so on with infinite terms. Actually, the ninth-order approximation (the first five non-zero terms in this case) is very accurate up to $x = \pi$. Try it.

The general process by which the series approximation terms are derived for a particular function is due to Brook Taylor, a mathematician who worked on what later became known as *differential calculus*. He published this, his most famous result, in 1712.

[6] Here's a simple geometric proof. If we consider a thin sector of a circle of angle θ and chord length l, we see that (*in radians*) we have $l = r\theta$. The perpendicular distance h is given by $h = r \sin\theta$. We see from the geometry of the sketch that for very small θ, $h \approx l$, and therefore $\sin\theta \approx \theta$.



[7] By n-th order we mean including terms up to x^n.

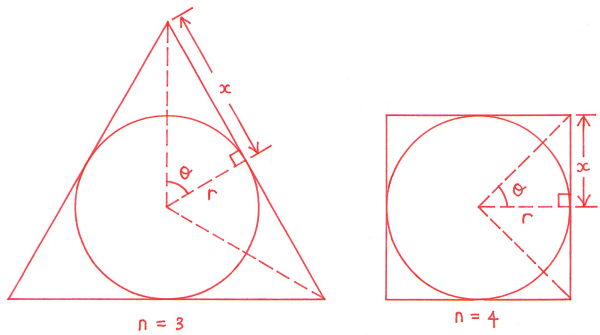
[8] Zero factorial is equal to unity. The proof is as follows. Noting that $(n+1)! = (n+1) \times n \times \ldots \times 2 \times 1 = (n+1) \times n!$, and setting $n = 0$, gives $1! = 1 \times 0!$, so $0! = 1! = 1$.

The Taylor series for $\cos x$ and $\tan x$ are as follows

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots \\ \tan x &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 - \ldots\end{aligned}$$

Answer

Consider the constructions for $n = 3$ and 4-sided polygons.



The polygons are regular, so using their symmetry we can decompose them into $2n$ right-angled triangles, each with area $(xr)/2$, where r is the radius of the circle, and x is the half-length of the side of the polygon, given by $x = r\tan\theta$, where $\theta = \pi/n$. The total area of the n-sided polygon becomes

$$A_n = nxr = nr^2 \tan\frac{\pi}{n}$$

Consider that

$$\frac{A_{shaded}}{A_{circle}} = \frac{A_n}{A_{circle}} - 1 = \frac{n}{\pi}\tan\frac{\pi}{n} - 1$$

So we require the minimum n that satisfies

$$\frac{n}{\pi}\tan\frac{\pi}{n} - 1 \leq \frac{1}{1000}$$

Here we must choose either a graphical, numerical (iterative) or series solution approach. Let us use the series solution, which is the most elegant, although trying the problem in other ways is also instructive. We need to approximate $\tan(\pi/n)$ using the Taylor series for $\tan x$. Taking a third-order approximation we have

$$\tan x \approx x + \frac{1}{3}x^3$$

Our inequality reduces to

$$\frac{n}{\pi}\left[\frac{\pi}{n}+\frac{1}{3}\left(\frac{\pi}{n}\right)^3\right]-1\leq\frac{1}{1000}$$

That is

$$\frac{1}{3}\left(\frac{\pi}{n}\right)^2\leq\frac{1}{1000}$$

This gives $n \geq \pi\sqrt{1000/3} = 57.4$. We require n to be an integer, so the minimum n that satisfies this condition is $n = 58$, an octapentacontagon, or, in the language most mathematicians use, a "58-gon".

Because we have used Taylor series approximations it is very important to check this solution directly, to see if it is correct. We do so by recalling the condition $(n/\pi)\tan(\pi/n) - 1 \leq 1/1000$, and evaluating this explicitly for $n = 57$ and $n = 58$. We get

$$\frac{57}{\pi}\tan\left(\frac{\pi}{57}\right)-1=1.014...\times10^{-3}\not\leq\frac{1}{1000}$$

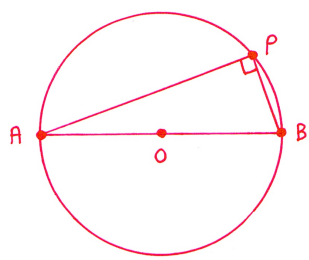
$$\frac{58}{\pi}\tan\left(\frac{\pi}{58}\right)-1=9.791...\times10^{-4}\leq\frac{1}{1000}$$

We see that this agrees with our result.

1.9 Triangle inscribed within semicircle ★

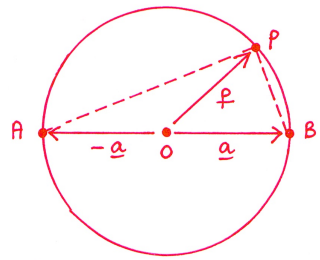
This little problem is an exercise in using vectors.

GIVE A VECTOR *proof* that for a triangle inscribed within a semicircle, the included angle is always $\pi/2$.

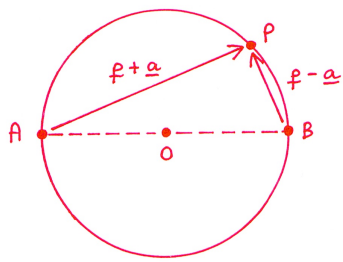


Answer

Consider the semicircle APB, formed by the intersection of a line AOB with a circle. From the centre of the circle, if B is represented by the vector[9] $\boldsymbol{a}$, then A is represented by the vector $-\boldsymbol{a}$. Let point P be represented by the vector $\boldsymbol{p}$.



The line AP is represented by the vector $\boldsymbol{p}+\boldsymbol{a}$, and the line BP is represented by the vector $\boldsymbol{p}-\boldsymbol{a}$. For two general vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, the dot product is defined by $\boldsymbol{A}.\boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}|\cos\theta$, where θ is the angle between the vectors. Thus, for $\theta = \pi/2$, $\boldsymbol{A}.\boldsymbol{B} = 0$. We recall two properties of the dot product. It is *commutative* (that is, $\boldsymbol{A}.\boldsymbol{B} = \boldsymbol{B}.\boldsymbol{A}$) and *distributive* (that is, $\boldsymbol{A}.(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}.\boldsymbol{B} + \boldsymbol{B}.\boldsymbol{C}$).



Consider the dot product of the two vectors defining AP and BP. Expanding using the distributive property we have

$$(\boldsymbol{p}+\boldsymbol{a})\,.\,(\boldsymbol{p}-\boldsymbol{a}) = \boldsymbol{p}.\boldsymbol{p} - \boldsymbol{p}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{p} - \boldsymbol{a}.\boldsymbol{a}$$

Simplifying using the definition of the dot product and the commutative property, we have

$$(\boldsymbol{p}+\boldsymbol{a})\,.\,(\boldsymbol{p}-\boldsymbol{a}) = |\boldsymbol{p}|^2 - |\boldsymbol{a}|^2$$

We note that both $\boldsymbol{a}$ and $\boldsymbol{p}$ are radii of the circle, so

[9] In this book we will use bold symbols to distinguish a vector $\boldsymbol{a}$ from its scalar magnitude $|\boldsymbol{a}|$, or a. In handwriting (the hand-drawn diagrams in this book, for example) vectors are denoted with underscores (such as $\underline{a}$).

$$|\boldsymbol{a}| = |\boldsymbol{p}|$$

We see that the dot product $(\boldsymbol{p}+\boldsymbol{a})\,.\,(\boldsymbol{p}-\boldsymbol{a}) = 0$. However, $|\boldsymbol{p}+\boldsymbol{a}| \neq 0$ and $|\boldsymbol{p}-\boldsymbol{a}| \neq 0$. So we conclude from the definition of the dot product that for $(\boldsymbol{p}+\boldsymbol{a})\,.\,(\boldsymbol{p}-\boldsymbol{a}) = 0$ to be satisfied we must have $\theta = \pi/2$. Thus, a triangle inscribed within a semicircle is always a right-angled triangle.

A friend who read this question pointed out, quite rightly, that easier solutions exist that don't require vector notation. He objected to the use of vectors on the basis that they over-complicate the analysis. He was right, of course, but I still think it's a nice exercise for those not entirely familiar with using vectors. I did feel obliged to include the alternative solution,[10] however.

1.10 Big and small tree trunks ★★ or ★★★★

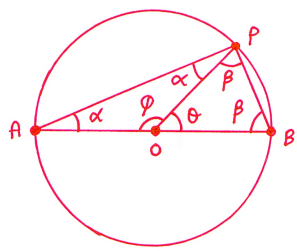
Quite a number of maths questions are designed to test a student's ability to set up equations for unfamiliar problems and then solve them. This is an extremely useful skill for a numerate person. The question here is one of several similar problems I came up with on a long-haul flight to the US a couple of years ago. The geometry is trivial, but I think most high-school students would struggle with it, simply because the ability to approach completely unfamiliar situations is valued too little in most curricula.

A TREE TRUNK has a circular cross section of radius n, where n is an integer. A second circular tree trunk has radius m, where m is also an integer. Both tree trunks rest on the ground, defined by $y = 0$, with their axes aligned with the z-axis.

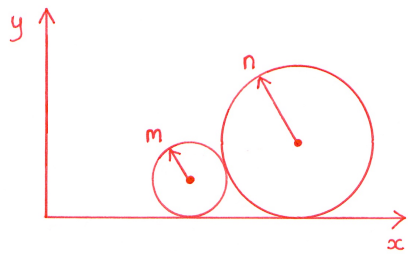
1. Derive the general condition that permits both tree trunk centres to lie at integer co-ordinates along the x-axis.

2. Give the three smallest solutions for n when $m = 1$.

[10] Noting the construction $APBOA$, we see that $\phi = \pi - 2\alpha$ and that $\theta = \pi - 2\beta$. Adding these expressions we get $\phi + \theta = 2\pi - 2\,(\alpha + \beta)$. But $\phi + \theta = \pi$. Combining these expressions we get $\pi = 2\pi - 2\,(\alpha + \beta)$. Simplifying, we get $\alpha + \beta = \pi/2$, as required. This solution is undeniably simpler.



3. Give a general solution for n and m (extension for ★★★★).

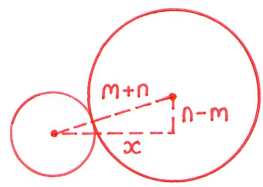


Answer

This is a simple geometry problem that I think a lot of students would struggle to unpack. Let's consider each of the three parts in turn.

1) DERIVE THE GENERAL CONDITION THAT PERMITS BOTH TREE TRUNK CENTRES TO LIE AT INTEGER CO-ORDINATES ALONG THE x-AXIS

We are told that n and m are integers, so the difference in the y co-ordinates of their centres is also an integer, $n - m$. We require that the difference in the x co-ordinate also be an integer. We denote the difference simply as x. We note that the tree trunks touch at a mutual point of tangency, so the distance between their centres is $n + m$.



From Pythagoras' theorem we have

$$(n+m)^2 - (n-m)^2 = x^2$$

Simplifying, we get

$$4nm = x^2$$

This is the general condition that must be met.

2) GIVE THE THREE SMALLEST SOLUTIONS FOR n WHEN $m = 1$.

A mathematician would hardly regard it as elegant, but to obtain the first three solutions for n when $m = 1$ we can solve for $m = 1$ simply by substituting this into the expression and trying values for n.

$$\text{for } m = 1,\ n = \frac{x^2}{4}$$

We can probably see the first three solutions to this. If we can't, we can simply try the first few integer values for x. We have

x	x^2	$n = x^2/4$	Integer?
1	1	$1/4$	N
2	4	1	Y
3	9	$9/4$	N
4	16	4	Y
5	25	$25/4$	N
6	36	9	Y

The first three solutions are $n = 1$, 4 and 9.

3) GIVE A GENERAL SOLUTION FOR n AND m (EXTENSION FOR ★★★★).

Consider the general condition that we are required to meet, namely $4nm = x^2$, or $2\sqrt{nm} = x$. We know that x is an integer. We write this as $x \in \mathbb{Z}$. Here $\in$ means *is a member of*, and $\mathbb{Z}$ represents the set of all integers. By extension $2\sqrt{nm} \in \mathbb{Z}$. It is possible to prove[11] that $\sqrt{k} \in \mathbb{Z}$ if and only if k is a square number.[12] From this it follows that there are two possibles cases:

- n or m is equal to unity and the other is a square number.
- $m = a^2b$ and $n = bc^2$, where b is not a square number and where a, b and c are integers.[13]

[11] PROOF THAT $\sqrt{k} \in \mathbb{Z}$ IF AND ONLY IF k IS A SQUARE NUMBER.

THEOREM: $\sqrt{k}$ is an integer if k is a square number, otherwise it is irrational (it cannot be expressed as the ratio of two integers). Here $k \in \mathbb{Z}^+$; that is, it is a member of the set of all positive integers.

PROOF: a) If k is a square number we write $k = l^2 \Rightarrow \sqrt{k} = l \in \mathbb{Z}$; b) We assume that $\sqrt{k}$ is a rational number, that is, $\sqrt{k} \in \mathbb{Q}$, and prove that this is impossible by contradiction. Consider two sets of numbers $\{a_i\}$ and $\{b_i\}$ in which members are positive integers: $a_i \in \mathbb{Z}^+$ and $b_i \in \mathbb{Z}^+$. We assume that we can write $\sqrt{k} = a_i/b_i$, where (a_i, b_i) is an ordered pair of positive integers. There must be a unique least element $a \in \{a_i\}$. Similarly, unique least element $b \in \{b_i\}$ exists. The least element of $\{a_i\}$ must pair up with the least element of $\{b_i\}$. Hence $\sqrt{k} = a/b$, so $k = a^2/b^2$, so $b^2k = a^2$. Because $a, b, k \in \mathbb{Z}^+$, and because the left-hand side is divisible by k, the right-hand side must also be divisible by k, so we can write $a = kc$ (provided that k is not a square number, otherwise we could write $a = \sqrt{k}c$), where $c \in Z^+$. We write $b^2k = (kc)^2 \Rightarrow \sqrt{k} = b/c$. But for $k \in \mathbb{Z}^+$ and $k \neq 1$, $\sqrt{k} > 1$. Therefore, from $\sqrt{k} = a/b$, we see that $a > b$. Similarly, from $\sqrt{k} = b/c$, we see that $b > c$. So we have found $\sqrt{k} = b/c$ with $c < b < a$. However, a and b were the least elements of $\{a_i\}$ and $\{b_i\}$ respectively, which leads to a contradiction. $\{a_i\}$ and $\{b_i\}$ must both be empty, so $\sqrt{k}$ cannot be written as a rational number. We have proved that $\sqrt{k} \notin \mathbb{Q}$.

[12] A *square number* or *perfect square* is an integer that is the square of an integer.

[13] TO SHOW THAT n AND m SATISFY THIS CONDITION. If $m = a^2b$ and $n = bc^2$, then $mn = a^2b^2c^2 = (abc)^2$. Where $a, b, c \in \mathbb{Z}$, then $abc \in \mathbb{Z}$, so $mn = (abc)^2$ is a square number.

Chapter 3

Statics

In this section we look at problems in *statics*. By definition, *statics* is the study of systems that are stationary, and non-accelerating. In these systems the forces acting on individual masses must sum to zero, and the *torques* (or *moments*) acting on a body must sum to zero. These are requirements for zero acceleration (both linear and rotational). We generally look at the systems at a particular instant, and *resolve*[1] the forces so that the conditions for zero acceleration are satisfied. We should state a few definitions more formally.

- Torque or Moment. The magnitude of the torque (or moment) about a particular point (or pivot) due to a force is the product of the force and the perpendicular distance between the line of action of the force and the point of interest. We refer to this distance as the *lever arm* of the force. Torque has units of N m.

- Equilibrium Requirement for Zero Linear Acceleration. For an object of finite mass, for the linear acceleration in a particular direction to be zero, the forces in that direction must sum to zero. For the simplest two-dimensional Cartesian system, with axes x and y, we write $\sum F_x = \sum F_y = 0$. This condition must hold in every direction.

- Equilibrium Requirement for Zero Angular Acceleration. Taking an object of finite moment of inertia[2] about a particular pivot, A, for the angular acceleration to be zero, the sum of torques, T_{Ai}, about that pivot must be equal to zero. We write $\sum_i T_{Ai} = 0$. By the same logic, given that every real body has a finite moment of inertia, the sum of torques about *any point* in a system must be equal to zero, whether or not that point lies on the body.

[1] To resolve, take the sum of vector components in a particular direction.

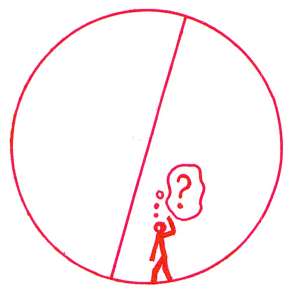
[2] In the same way as mass is a measure of the resistance of a body to linear acceleration, the moment of inertia is a measure of the resistance of a body to angular acceleration. The moment of inertia I of a body is defined about a particular point. For a body composed of mass elements m_i at distances r_i from a point of interest A, the moment of inertia is defined by $I_A = \sum_i m_i r_i^2$.

- CENTRE OF MASS (GRAVITY). The centre of mass (or gravity) of an object is defined as the point through which any plane drawn would split the object into two parts of equal mass. It can be thought of as the point from which the weight of the body acts. It does not necessarily need to lie on the object (consider a doughnut shape, for example).

Now we are ready to try a few problems in statics. Most of these questions are my own invention, but a couple are modifications of well-known problems.

3.1 Sewage worker's conundrum ★

A SEWAGE WORKER is inside a large underground aqueduct (diameter $\gg$ a man's height) of circular cross section, and so smooth that friction is negligibly small. He has a ladder which is the same length as the diameter of the aqueduct. He wishes to inspect something in the roof. He mounts the ladder and continues to climb until he reaches the other end. What happens?



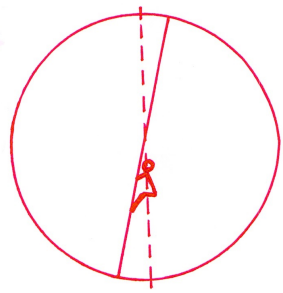
Answer

It hardly matters as far as solving the problem is concerned, but we might start with the realization that the sewage worker must be at the lowest point of the aqueduct. Were he anywhere else he would slide to the bottom because the aqueduct is perfectly smooth. The ladder, interestingly, is equally happy in any rotational position despite there being no friction. We could argue the case by drawing forces on the ladder. Or we could simply note that it has the same potential energy in all rotational positions. This must be so because the centre of mass of the ladder is always at the centre of the tunnel: the length of the ladder is equal to the diameter of the tunnel. Energy arguments can be powerful, and we will use them in this question.

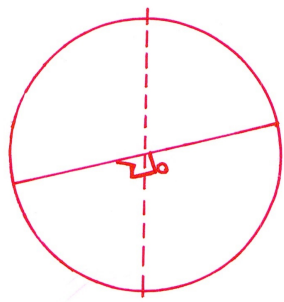
But what *does* happen next?

Well, as the man takes hold of the ladder and mounts it, the very first thing that happens is that the ladder tips back slightly. Because the *system* is free to move, it moves to the position with the lowest potential energy. Here we are

using the *principle of minimum total potential energy*.[3] When we talk about the system, we mean the ladder and the man. The potential energy of the ladder is the same in every rotational position. The man's potential energy varies with the angle he makes with the vertical, and is minimised when his centre of mass is below the centre of the ladder, which can be thought of as a virtual pivot.[4] The man's centre of mass hangs slightly away from the plane of the ladder, and his centre of mass must be under the centre of rotation of the system for it to be in equilibrium.



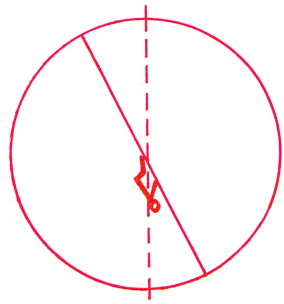
As he climbs and gets closer to the halfway point of the ladder, which is the centre of rotation of the system, the ladder has to tip back more. When the man approaches the middle of the ladder, the ladder must be almost horizontal, until his weight is directly under the centre of the cylinder and the ladder *is* horizontal.



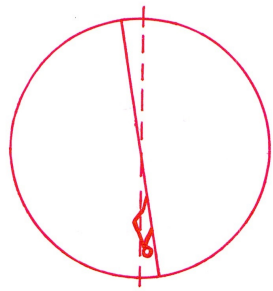
[3] This rather impressive sounding principle simply tells us that systems that are free to move do so until they are at the point of minimum potential energy. The *lost* energy is dissipated, as heat, for example. A ball placed on the top of a hill will roll to the bottom of the hill. Water will move until the free surface is perpendicular to the direction of gravity. The most stable configuration is the one with minimum potential energy. This is the *equilibrium position* for the system.

[4] If this is not perfectly obvious, consider that when the man is a given distance, say r, from the midpoint of the ladder, his possible positions (for every angle θ his centre of mass makes with respect to the vertical) are defined by a circle of radius r centred on the midpoint of the ladder. The lowest point on that circle, or the point with the minimum potential energy, is below the midpoint of the ladder.

As the man continues to climb along the ladder from the midpoint, the ladder rotates past the horizontal position. The man climbs down to the bottom of the cylinder, this time upside-down.

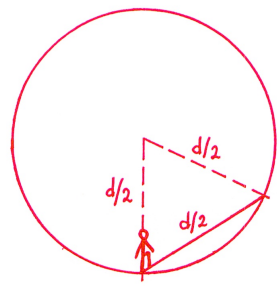


Eventually he is back to where he started but the wrong way up. It seems his inspection of the roof will have to wait.



3.2 Sewage worker's escape ★★

A SEWAGE WORKER is inside a large and smooth underground aqueduct of circular cross section and diameter d. To escape a tide of effluent, he balances on the end of a ladder of length $d/2$ and mass m. If the sewage worker also has mass m, at what angle does the ladder settle? Assume that the length of the ladder is very much greater than the height of the man, so he acts as a *point mass* on the end of the ladder.

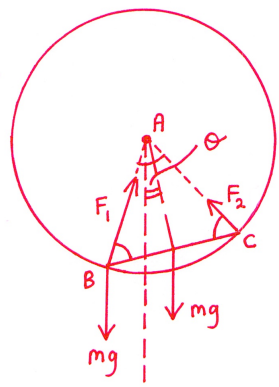


Answer

We first need to consider the geometry of the problem. Triangle ABC is equilateral, with side $d/2$. We might spot at this point, however, that the answer is unlikely to depend on $d/2$, as it is determined by the angles of the system. The angle at which the ladder settles we mark as θ in the diagram. There are four external forces acting on the ladder-man system.

- The weight due to the ladder, mg, which acts vertically downwards from the centre of the ladder.
- The weight due to the man, mg, which acts vertically downwards from point B (we treat the man as we would a *point mass* at B).
- The normal reaction force F_1, which acts perpendicular to the smooth wall at point B.
- The normal reaction force F_2, which acts perpendicular to the smooth wall at point C.

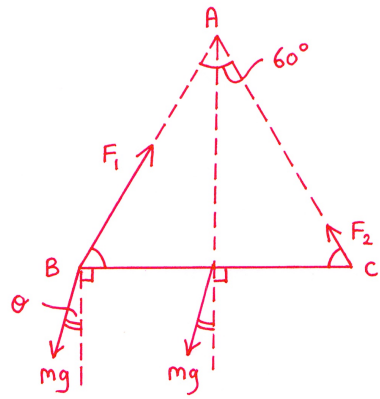
Now we have the four forces we can mark them on the diagram, paying attention to the directions in which they act. We note that forces F_1 and F_2 both act through the centre of the cylinder, along lines BA and CA respectively.



For the man-ladder system to be in static equilibrium, the forces in all directions have to balance. If we had a net force in a particular direction we would have a non-zero acceleration in that direction according to Newton's Second Law. By similar reasoning we require zero net torque (moment) on the man-ladder system, so that it is in rotational equilibrium.[5] We are interested in the

[5]Netwon's Second Law tells us that the force in a particular direction $\boldsymbol{F}$ (where we note this is a vector) is equal to the time rate of change of momentum in that direction, $\boldsymbol{F} = d\boldsymbol{p}/dt$. For constant m we have $\boldsymbol{F} = m\,(d\mathbf{v}/dt) = m\mathbf{a}$. To be in static equilibrium we must have zero acceleration, $d\mathbf{v}/dt = \mathbf{a} = \mathbf{0}$. For this we require the sum of forces in every direction to be zero. To be in static equilibrium we also require the angular acceleration to be zero: $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt = \mathbf{0}$, where $\boldsymbol{\omega}$ is the angular velocity vector. For this we require the net torque on the object to be zero, giving $\boldsymbol{\tau} = I\boldsymbol{\alpha} = 0$.

vector forces, and the points of action of the forces, on the man-ladder system. As long as we preserve the relative magnitudes, angles, and distances we can consider the diagram in any rotated position. A slightly simpler representation of the man-ladder system is shown below. Note that the weight vectors make an angle θ to the ladder, as they did in the original representation.



The standard procedure from here, which should be the mantra of any student of statics, is to *resolve, resolve, take moments*. Let us do just that.

- Resolve in the horizontal direction: $2mg\sin\theta = F_1\cos 60° - F_2\cos 60°$
- Resolve in the vertical direction: $F_1\sin 60° + F_2\sin 60° = 2mg\cos 60°$
- Take moments about point B: $2F_2\sin 60° = mg\cos\theta$

Now we just need to solve these equations for θ, the angle of the ladder to the horizontal. We should check that we have enough equations: we need as many independent equations as we have unknowns. The unknowns are F_1, F_2 and θ—only three. We have three independent equations, so a solution should be possible. I've left the algebra as an exercise, as it should be fairly straightforward from here. If you do it correctly you should get

$$\theta \approx 16.10°$$

The answer feels about right: the ladder is at a shallow angle to the horizontal. Of course, if we wish, we can then also solve explicitly for F_1 and F_2, giving

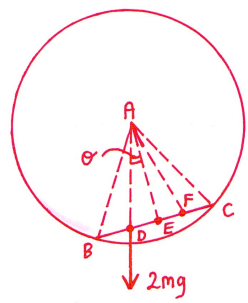
$$\begin{aligned} F_1 &= \sqrt{3}mg\cos\theta \approx 1.66\,(mg) \\ F_2 &= \frac{\sqrt{3}}{3}mg\cos\theta \approx 0.555\,(mg) \end{aligned}$$

Again, the answers feel about right. F_1 is approximately three times larger than F_2, giving greater reaction force at the end where the man is balancing, as we'd expect.

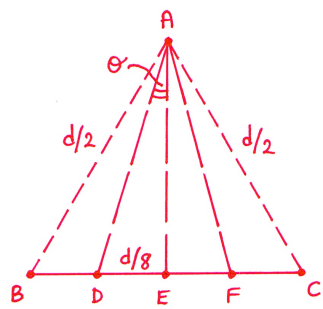
Further discussion

We did well to solve the last problem. We took a completely new situation, reduced it to the underlying equations, then solved the equations quite simply to get an exact result. The *resolve, resolve, take moments* method worked well. But there are many other ways to tackle problems of this sort, and here we demonstrate one that avoids the need to explicitly deal with forces and moments. It is a very common method in statics problems, and a very common technique in studying *systems* in general. It is to consider the *energy* of the system—specifically, the point of minimum potential energy for the system comprising the ladder and the man. Many systems that are free to move to a lower potential energy point *do* move to that point. This is one such system.

The lowest potential energy state for this system is the one in which the centre of mass is as low as possible. The centre of mass of the system is midway between point B and point E, or $1/4$ of the way along the ladder from point B. The lowest potential energy state is when point D is directly below point A. If this is not obvious, consider varying θ between $0°$ and $360°$ in the diagram. Point D describes a circle about point A. It should now be obvious that the lowest point of the circle is directly below A.



It is now a simple exercise in geometry to define the angle θ that line AE makes to the vertical. This is the same angle that BC makes to the horizontal. Consider the geometry of the situation.



We have

$$\theta = \arctan \frac{(d/8)}{|AE|} = \arctan \frac{1}{2\sqrt{3}} \approx 16.10^\circ$$

This is the same answer as before. By using the principle of minimization of potential energy we have considerably simplified the problem.

Assumptions

We started by saying that the aqueduct was smooth, and we implicitly suggested that the man could get onto the ladder in a position that was not the point at which the ladder ultimately settles. If we wanted to be pedantic, we could argue that if there was no friction *at all* the ladder would simply oscillate indefinitely, and never settle. Likewise, getting onto the ladder could be problematic. If the expected answer to "at what angle does the ladder settle?" was either of these I think we would feel we had been somewhat misled. We therefore assume that the walls are *smooth enough* that the reaction forces are always normal to the wall, but not quite so smooth that the ladder cannot eventually settle.

An important skill in addressing physics problems is identifying the fundamental subject of interest, the nub of the question, and filtering out everything that is unimportant. In that latter category are an almost infinite number of things which we should know we need to discount: the material of the ladder—we know it is smooth; air resistance—we are considering an equilibrium/stationary solution; the name of the man; the day of the week; etc... If we feel the need to introduce these latter things, we are probably more interested in riddles than physics problems.

3.3 Sewage worker's resolution ★★★

A SEWAGE WORKER needs to use a ladder inside a large and perfectly smooth underground aqueduct of circular cross section. The ladder is of the same length as the diameter of the aqueduct. Draw the forces on the ladder when the ladder is placed: i) vertically; ii) horizontally. Discuss the diagrams.

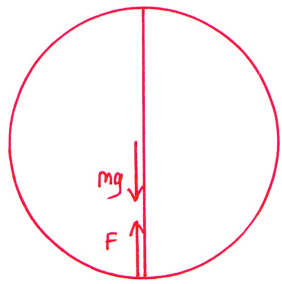
Answer

We consider the answer in two sections. First the vertical ladder, then the horizontal ladder.

i) VERTICAL LADDER

This should seem, and is, very simple. There are only two external forces acting on the ladder:

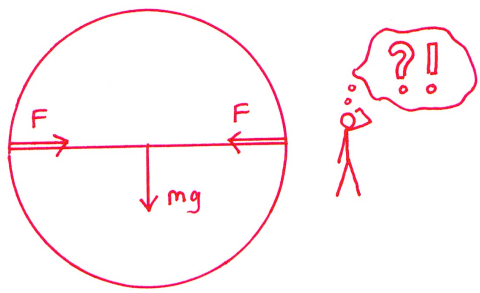
- The weight of the ladder, mg, which acts vertically downwards from the centre of mass of the ladder.
- The reaction force, F, which acts normal to the smooth wall of the cylinder at the point of contact.



If the ladder is in static equilibrium the two forces must be in balance. That is, $F = mg$.

ii) Horizontal Ladder

Our first reaction might be that this is equally simple. We can have three external forces acting on the ladder, one due to the weight of the ladder, and one at each end of the ladder. The ladder is symmetrically disposed about the y-axis, so we expect the two forces at the ends of the ladder to be equal in magnitude and opposite in sign. We know that the wall of the cylinder is smooth, so the forces at the ends must be normal to the wall of the cylinder. The arrangement of forces is shown in the diagram below.

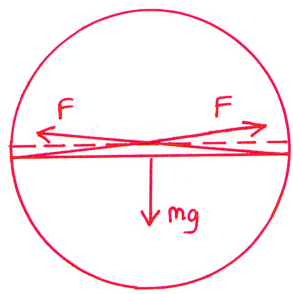


A problem is immediately apparent. We can balance forces in the horizontal direction, but what about the vertical direction? We seem to have a net downward force due to the weight, mg. The ladder cannot be in static equilibrium with this arrangement of forces. We require upward forces at the walls to balance the weight of the ladder. As the boundary conditions are stated, and, making assumptions about the ladder having negligible width in the vertical direction, it is impossible to achieve a balance of forces.

There are a number of ways we can modify the boundary conditions, or relax the assumptions, that would allow us to have a more realistic arrangement of forces.

- We could relax the smooth wall condition, and have a coefficient of friction slightly greater than zero.
- We could allow the ladder to have finite width in the vertical direction, so that the points of contact are just above and just below the centre of the cylinder. This would allow an upward component of reaction force at the lower points of contact.
- We could relax the condition that the ladder is exactly the same length as the diameter of the cylinder, and allow it to be infinitesimally shorter. This would allow it to drop a small distance below the centre of the cylinder, allowing an upward component of force at the points of contact. In practice this would happen due to both bending and compression.

Let us consider the option of the ladder being infinitesimally shorter. This could be on account of a tiny amount of compression, or due to very slight bending, or both. Whatever the mechanism, the points of contact with the cylinder move just below the line through the centre of the cylinder. If we still make the assumption that the cylinder is perfectly smooth, the symmetrical reaction forces, F, are still normal to the wall, and are therefore directed through the centre of the cylinder. This arrangement of forces allows us to have zero net horizontal and vertical force on the ladder. The conundrum is resolved. It is interesting to note that for a small amount of compression we require $F \gg mg$, so the mechanism for providing measurable compression of even a fairly rigid ladder is also apparent.



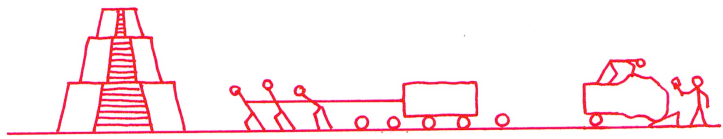
3.4 Aztec stone movers ★★

There is some uncertainty about how the Aztecs and Egyptians moved huge stones great distances to their temples and pyramids, even if we assume that a practically unlimited supply of slaves must have been available. There is even

more uncertainty about how our Neolithic ancestors behind such structures as Stonehenge could have accomplished what even today would have been quite a feat of engineering. The sarsen stones that form the outer circle of the monument weigh 50 tonnes each. To lift such a stone would take the strength of about 1,000 men—unfortunately only about 20 men fit round the perimeter of each stone. The nearest known source of sarsen stone is 25 miles from the monument. And the stones were transported in about 2000 BC.

We have almost no direct evidence of how such a feat was accomplished, but most theorists have it that the stones were moved using sleds and ropes. The sleds were made of tree trunks, and the ropes of leather. The sleds were loaded onto rollers, also made from tree trunks. It has been estimated that using this relatively sophisticated system would have required 500 men to pull the load and a further 100 men to place the rollers.

IF A BLOCK of stone 2 m long is to be pulled 1 km, how many times do rollers have to be placed in the path of the stone?

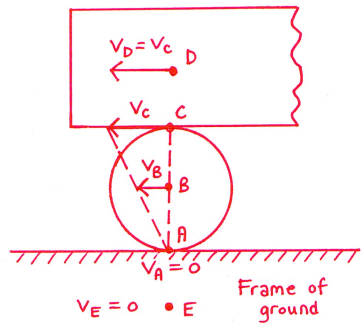


Answer

The *obvious* answer to this goes as follows. The block of stone is 2 m long and can only balance on a single roller when that roller is exactly in the middle of the block. At this point the trailing roller could be removed and a leading roller could be added. The separation of the rollers is therefore 1 m, and approximately 1,000 rollers would have to be placed along the 1 km path. As you have already guessed, this answer is wrong. That would be too easy to be interesting.

The correct solution is only a little more complicated, however. The key observation lies in the fact that a circle rolling along the ground is instantaneously stationary at the point of contact. If it were not, slipping would occur. Consider points A, B and C in the diagram below. Point A is the point of contact with the ground. It is the point about which—instantaneously—the body appears to be rotating. We conclude that $v_A = 0$. Neither is the ground moving, so $v_E = 0$. Points B (the centre) and C (the top) lie vertically above point A. Thus, B and C can have *only horizontal components of velocity* if we consider that the body is rotating instantaneously about A. This is an important point, which is worth thinking about. If the wheel is in solid-body rotation about point A, the velocity of C must be twice the velocity at B, so $v_C = 2v_B$. This is easy to see because B and C are, respectively, one radius

and two radii from A. The velocity distribution for points on a line between A and C is linear; that is, a triangle of velocities is formed.[6]



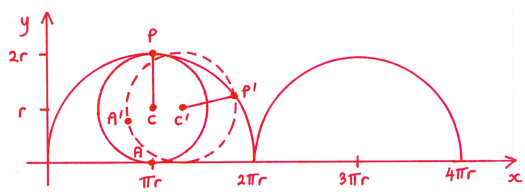
We can apply a similar argument to the block of stone. If the roller *rolls* against the block of stone, rather than slipping, the roller and the stone must have the same velocity at the point of contact, so $v_D = v_C$. The important result is that the speed of advance of the stone is twice the speed of advance of the roller. We will return to this in a moment.

For interest's sake, let's consider the problem in the frame of reference of the block, that is, the frame of reference in which the block appears to be stationary, $v'_D = 0$. This is the frame we would be observing from if we were sitting on top of the block as it was being slid along by the workers. In our *block frame* we will mark all velocities v', to distinguish them from velocities measured in the *ground frame*, v. In our block frame it looks like the ground is being slid along on rollers underneath the block. The ground has to go in the opposite direction to the block when observed from this frame. Because there is no slipping at the point of contact with the block, $v'_C = v'_D = 0$.

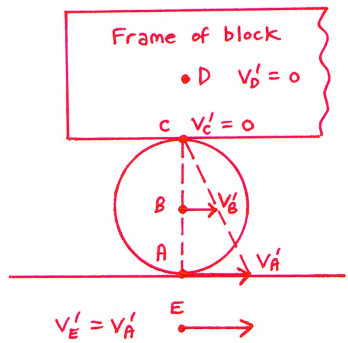
[6] A popular, related puzzle is how to derive the equation of a *cycloid*, or the path traced by a point on the rim of a wheel as it rolls without slipping along a plane. For a wheel of radius r, which has rolled about its centre through an angle θ, it is possible to show using relatively simple geometry that the *parametric equations* (that is, the equations for x and y as a function of θ) are $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$. By eliminating θ it is possible to show that the Cartesian equation ($x = f(y)$) is given by

$$x = r\cos\left(1 - \frac{y}{r}\right) - \left(2ry - y^2\right)^{1/2}$$

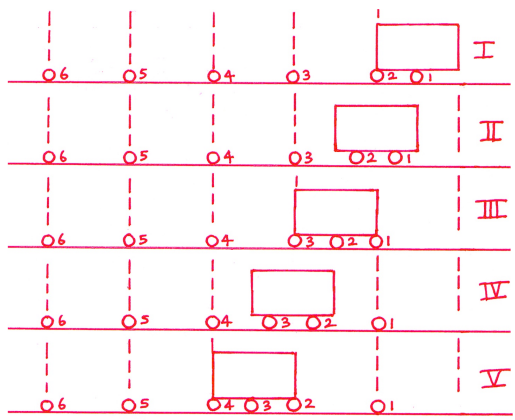
This is valid for $y = 0$ to $y = 2r$, and gives the first half of the first hump (from the origin to P in the diagram).



Once again, the velocity distribution along a vertical line of points through the centre of a roller is linear in form, giving $v'_A = 2v'_B$. In this frame of reference, the ground is moving twice as fast as the centre of the roller. Because there is no slipping at the ground, $v'_E = v'_A$. There are no great surprises here, but it can sometimes be helpful to look at problems like this from another perspective to confirm that they still make sense.



The key result of this analysis is that for every 1 m the block is pulled forward, the roller advances 1/2 m. We decided at the outset that we needed the rollers to be spaced exactly 1 m apart under the block. But this would be achieved by having them spaced every 2 m apart on the ground, because as the roller is pulled forward 2 m, the rollers already underneath the block go back only 1 m relative to the block. The process is illustrated below.



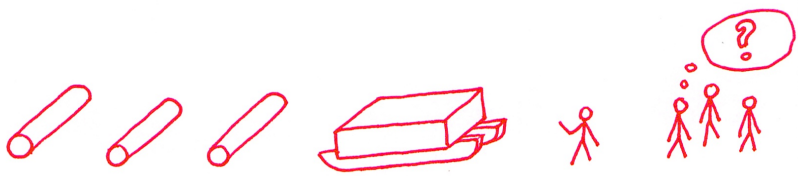
Further discussion

It is interesting to note that the answer is independent of the diameter of the rollers. Note that we did not specify the diameter in the analysis. In fact, as we will see in questions relating to imperfect wheels, the force required to

rotate an imperfect wheel on a smooth surface also depends only on *shape*,[7] and does not depend on diameter. When rolling on a rough surface, however, the diameter of the wheel is important, and on rough surfaces larger wheels, or larger rollers, are clearly better.

There is a convincing case that the development of wheels was inspired by observations about the relative speed of rollers and load. In this version of history, men started with the observation that pushing or pulling great loads was made easier by using rollers made from tree trunks.

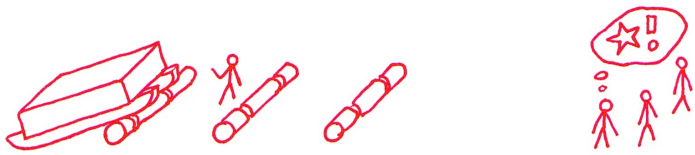
Around the same period in man's technological development, people started to use primitive sleds to haul loads, because the smooth runners had lower friction on most surfaces than the uneven loads did. One day, a particularly clever person realized that combining the sled and the roller was more efficient than using either in isolation.



Many many years passed in which the roller-sled system was adopted more routinely for heavy loads. When objects were hauled very long distances the sleds cut into the rollers, creating deep grooves in them. This served to guide the sled, which was helpful, but there was an even more important effect. As the grooves got deeper, the rollers needed to be placed less frequently. With very deep grooves you might need to place the rollers, say, ten times less frequently than with un-grooved rollers.[8] This made moving loads over long distances much more efficient.

[7] By *shape* we mean how circular the wheel is. In a question that follows, we model imperfections in the wheel, or facets, by representing the wheel as an n-sided polygon.

[8] We require a diagram to explain this interesting effect fully, but I've left that as an exercise for the reader. Consider, however, that when the grooves get very deep, the radius r of the remaining axle of wood (at the very bottom of the groove, that is) is very much less than the radius of the roller, R. We write $r \ll R$. We should be able to show that the forward progression of the load in the lab frame is proportional to $R + r$, and the backward progression of the roller with respect to the load (in the load frame) is proportional to r. The distance we can travel before having to replace a log is therefore proportional to $(R + r)/r$, and goes up quickly as r becomes smaller. Prove it for ★★★!



Neolithic man then started pre-grooving the rollers, creating what we would now recognize as axles with fixed wheels, which were placed periodically under the load. Later a single axle with fixed wheels was attached through fixed holes in the sled. Now there was relative motion of the bearing and axle, so the holes had to be lubricated with animal fat. From here it was only a small step to design a fixed (non-rotating) axle with removable wheels.

And the rest, as they say, is history.

3.5 The Wheel Wars I ★★★

About five thousand years ago, tens of thousands of years of somewhat uncertain prehistory were ending rather abruptly in almost all corners of the world, with the almost simultaneous beginnings of Copper Ages, Bronze Ages and Iron Ages. The historical evidence suggests that smelting techniques were first used about 5,500 years ago in Pločnik, modern day Serbia. And by 2000 BC, a mere 3,500 years later, metallurgy had spread to South America, where it was used in Pre-Inca cultures to extract gold and silver. This comparative rush of technological development marked the end of the Neolithic Era, which was itself the end of countless millennia we call the Stone Age. It is strange to think that our human ancestors had developed such a high degree of latent sophistication, which would go on to suddenly manifest itself in all the great inventions during the few thousand years between then and now.

One of the most important inventions in human history, along with the tools to make fire (fire itself had been around for a bit longer), and much later the printing press and penicillin, was the wheel.

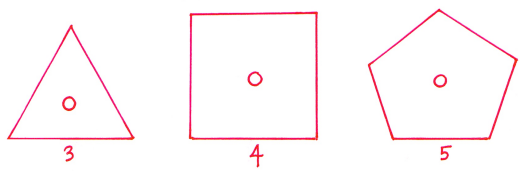
The first wheels seem to have been invented at similar times in several places: Mesopotamia (a region between the Tigris and Euphrates rivers which includes a large part of modern-day Iraq), the Indus Valley (modern day Afghanistan and Pakistan), the North Caucasus (home to the Maykop culture) and Central Europe. The very oldest intact wheel is thought to be the Ljubljana Marshes Wheel, which is believed to be 5,150 years old. Some people believe that the oldest known representation of a wheeled vehicle is on the Bronocice pot, dug up in the 1970s in Poland, and which dates back to 3500 BC.

As with all interesting topics in history, there are naysayers, those who believe the repeated motif is not a cart at all. You can decide for yourself. From around 3500 BC onwards, however, there is increasing evidence that people were using the wheel for pottery-making, transportation, and many other applications besides. The fanciful cartoon images of cavemen with stone wheels are just that. Strangely, however, huge stone wheels were, and to some extent still are, used on the island of Yap (in Micronesia) as a form of currency. The wheel truly has wide application.

The wheel's ability to roll is a strong function of how *smooth* it is—how far it deviates from a perfect circular form. Of course, it also depends on how flat the road is—a problem in ancient times.

During the Wheel Wars, a very brief period of Paleolithic history between 1,000,000 years ago and 1,000,001 years ago—the exact dates are disputed by some scientists—there was a sudden surge in wheel research, and wheel makers were selling wheels with three, four, and five sides, or even special made-to-order n-sided wheels. Wheel sellers claimed that it was worth spending more caveman currency (probably mammoth tusk or some such) on wheels with more sides, and it certainly seemed to be the case that large-n wheels rolled very smoothly indeed. But it would be another million years until mathematics was advanced enough to prove or disprove these marketing claims.

How much horizontal force, F, applied at the frictionless axle of a regular n-sided polygon of mass m, is required to cause it to roll without slipping?[9] If we wish, we can take r to be the distance from the axle to the vertices of the polygon, defined so that in the limit $n \to \infty$, r defines the radius of a circular wheel.



Answer

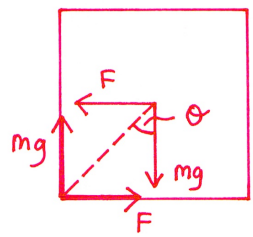
It may have been beyond the average Neolithic man, but this is actually relatively easy to answer. Let's start with a polygon with an arbitrary number of sides: $n = 4$, say (a square). In a moment we will see that it is a trivial extension to generalize the answer.

First we consider the point about which the square rotates. If we apply a force towards the left, and if the square rolls without slipping, the point about which the square rotates is the bottom left corner. When the object is on the point of overbalancing—that is, when it is about to rotate—all the force from

[9] This implies that we must assume that the coefficient of friction is high enough that this motion can occur.

the ground is transmitted through the point of rotation. If this is not obvious, consider the object a moment later.

If we apply a horizontal force F at the centre of the square (the axle), it must be balanced by a force equal in magnitude and opposite in sign which acts at the point about which the square rotates. Similarily, the downward force due to weight, mg, must be balanced by an upward force, mg, acting at the point of rotation. These two conditions are required for the object to be in static equilibrium—in other words, not accelerating in either the horizontal or vertical direction. We can now take moments about the point of rotation. When the applied force, F, is just large enough to cause rotation to occur, the moments about the pivot due to this force, and the weight mg, must be equal and opposite.



We have

$$Fr\cos\theta = mgr\sin\theta$$

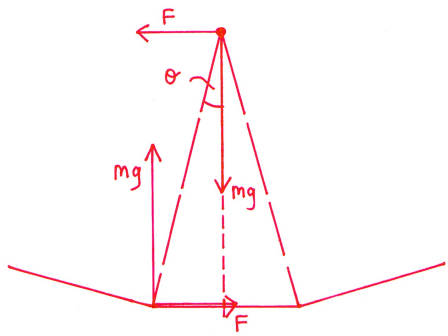
where r is the distance from the axle to the vertices of the polygon. We can rearrange this to give

$$F = mg\tan\theta$$

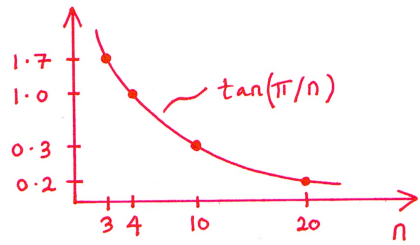
where, for a square, $\theta = 2\pi/8$. In general for an n-sided polygon, we have $\theta = 2\pi/2n = \pi/n$. So the required force becomes

$$F = mg\tan\left(\frac{\pi}{n}\right)$$

Because the side of a polygon subtends a smaller angle at the axle as n increases, the required force reduces rapidly as n increases. As $n \to \infty$, $\theta \to 0$, so the required force also approaches zero. A perfectly circular wheel (with no friction at the axle) requires no force to rotate on flat ground. A million years later we have proved the cavemen right—it was worth spending a bit more mammoth tusk on a wheel with lots of sides.



If we plot the solution we see that for $n = 4$ the force required to rotate the object is equal to mg. For a twenty-sided polygon $F = 0.16mg$. The solution *asymptotes* to zero for $n \to \infty$, which represents a perfect circle.



Further discussion

You will probably have noticed that in the analysis we have just performed the solution for the force required to rotate a wheel does not depend on diameter. Yet we know from practical experience that larger wheels give a smoother ride than smaller wheels. Bikes are less shaky than push scooters. There are a number of things at play here.

- Imperfections in the flatness of the ground act in the same way as facets on the wheel, and lead to large starting forces, and, by the same argument, a bumpy ride.
- Manufacturing tolerances become proportionally more important as wheels become smaller.
- Larger wheels often have additional mechanisms to reduce the peak force. These could include pneumatic tyres, or some form of suspension, or just the *compliance*[10] of the overall structure of the wheel.

[10] *Compliance* is the inverse of stiffness, or the deformation of a structure under a given load. The compliance of a structure is often responsible for the redistribution of stress, so that peak stress within the structure is reduced.

Chapter 4

Dynamics and collisions

In this chapter we look at dynamics and collisions. We consider collisions in terms of the velocities of the interacting bodies, the forces on the bodies during a collision, and the energy of the bodies before and after a collision. In the simplest case, we deal with head-on collisions—that is, one-dimensional collisions. The mathematics for two- and three-dimensional collisions is not much more complicated, but we need to consider the vector components (in the x, y and z directions, or in polar co-ordinates) when solving the equations. The basic principles remain the same.

To solve problems in dynamics and collisions we need a number of concepts to hand. We won't elaborate on all the equations here, because you should be able to *derive* them as you solve these problems. If we have the basic principles correct, however, the rest should follow naturally. Let us consider some of the concepts that come up in dynamics problems, and the notation that is used. If any of these are unfamiliar to you, it may be worth consulting a textbook before trying the problems.

- VELOCITY. When we consider interacting bodies, we are generally interested in their *initial* and *final* velocities—that is, the velocities before and after the collision. We need a notation to distinguish these velocites. For example, we can use $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ for the initial velocities, and $\boldsymbol{v}'_1$ and $\boldsymbol{v}'_2$ for the final velocities. The corresponding scalar *speeds* are v_1, v_2, v'_1 and v'_2. In general we will use the prime symbol ($\prime$) to indicate the state following a collision.

- ENERGY. The kinetic energy of a body is given by KE $= \frac{1}{2}mv^2$ (see proof[1]).

[1]This proof of the formula for kinetic energy is a popular problem that encourages students to think more deeply about an equation we normally take for granted. The incremental work dW performed on a particle is defined as the force F acting on the particle multiplied by the displacement in the direction of the force ds. In vector notation we write $W = \int_{s1}^{s2} \boldsymbol{F}.d\boldsymbol{s}$, where we take the dot product of vectors $\boldsymbol{F}$ and $\boldsymbol{s}$. If we consider motion in a single direction—the

- LINEAR MOMENTUM. The momentum of a body is given by $\boldsymbol{p} = m\boldsymbol{v}$. Because velocity is a vector quantity, momentum must also be a vector quantity. According to the law of conservation of momentum, in all collisions, whether elastic or inelastic, the linear momentum of the system is conserved. For two bodies we write

$$\begin{aligned} m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 &= m_1'\boldsymbol{v}_1' + m_2'\boldsymbol{v}_2' \\ \boldsymbol{p}_1 + \boldsymbol{p}_2 &= \boldsymbol{p}_1' + \boldsymbol{p}_2' \end{aligned}$$

- ELASTIC AND INELASTIC COLLISIONS. In an *elastic* collision the total energy of the bodies is *conserved*: the sum is the same before and after a collision. In an *inelastic* collision energy can be lost. By *lost* we mean that it is converted from kinetic energy, which is the useful form that interests us, to less useful forms of energy like heat.[2] For two bodies we write

$$m_1 v_1^2 + m_2 v_2^2 \geq m_1 \left(v_1'\right)^2 + m_2 \left(v_2'\right)^2$$

  where the equality applies for elastic collisions.

- COEFFICIENT OF RESTITUTION. The coefficient of restitution e is defined as the ratio of relative velocities after and before a collision. We write

$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = -\frac{|\boldsymbol{v}_1' - \boldsymbol{v}_2'|}{|\boldsymbol{v}_1 - \boldsymbol{v}_2|}$$

  where, in general,[3] $0 \leq e \leq 1$. For an *elastic* collision $e = 1$, for an *inelastic* collision $e < 1$, and for a *perfectly inelastic* collision $e = 0$. Later in this chapter we prove that, for elastic collisions, saying there is zero kinetic energy loss is the same as saying that $e = 1$.

- IMPULSE AND FORCE. Newton's Second Law tells us that force on a body is equal to the rate of change of momentum of the body. We write $\boldsymbol{F} = d\boldsymbol{p}/dt$, where the momentum $\boldsymbol{p} = m\boldsymbol{v}$. This reduces to

---

x-direction, say—this reduces to $W = \int_{x1}^{x2} F_x dx$. The acceleration of a particle of mass m is related to the force F_x by Newton's Second Law, $F_x = ma_x$. We write $W = \int_{x1}^{x2} ma_x dx$. The acceleration a_x is defined in terms of the velocity v_x as $a_x = dv_x/dt$. We write

$$W = \int_{x1}^{x2} m\frac{dv_x}{dt}dx = \int_{x1}^{x2} m\frac{dv_x}{dx}\frac{dx}{dt}dx = \int_{x1}^{x2} m\frac{dv_x}{dx}v_x dx$$

Simplifying, we write $W = \int_{v1}^{v2} mv_x dv_x$, where we change the limits to be in terms of velocity, the variable in which we are integrating. Integrating we get

$$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

which we define to be the kinetic energies of the initial and final states.

[2] One measure of the usefulness of a particular form of energy is the ease with which it can be converted to another form. In this context kinetic energy can readily be converted to heat, but heat cannot as readily be converted to kinetic energy.

[3] For *superelastic* collisions, or collisions in which stored energy is released, we can have $e > 1$.

$$\boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt} + \frac{dm}{dt}\boldsymbol{v} = m\boldsymbol{a} \quad \text{for} \quad \frac{dm}{dt} = 0$$

We can integrate this relationship to get the impulse on a body due to a force, which is simply defined as the integral of the force over a period of time. So, impulse $\boldsymbol{I}$ is defined by

$$\boldsymbol{I} = \int_t^{t'} \boldsymbol{F}dt = \int_p^{p'} d\boldsymbol{p} = \boldsymbol{p}' - \boldsymbol{p}$$

If the force is constant during the period of time over which we are integrating, we get

$$\boldsymbol{I} = \boldsymbol{F}\Delta t = \Delta\boldsymbol{p} = \boldsymbol{p}' - \boldsymbol{p}$$

So to achieve a particular change of momentum of a body—to bring it to rest, for example—we require a fixed impulse. The magnitude of the force required to achieve that impulse depends on the period of time over which the force acts. If we fall over on a concrete floor, in the resulting collision we are brought to rest very quickly indeed and the force is high. It hurts. If we fall over and hit a springy surface we are brought to rest rather slower and the peak force is lower.[4] In each case the total impulse is the same.

- Reference frame. According to the principle of *Galilean invariance*, Newton's laws apply in all inertial frames of reference. An inertial frame of reference is one for which the acceleration is zero. This means that the laws of conservation of momentum and energy (for elastic collisions) apply equally in any non-accelerating frame. In 1632 Galileo used the example of an observer in the belly of a ship sailing over a smooth sea. According to the invariance principle, when performing experiments the observer would be unable to detect whether the ship was in constant motion or stationary. For all practical purposes this statement is true, but we should note that the Earth's surface is not quite an inertial frame. This is because it rotates about its own axis, and about the Sun, which itself rotates around the centre of the galaxy, and so on. We will ignore the secondary non-inertial effects, which are small.

  In addition to the laboratory frame of reference, which we take as being the frame in which the observer is stationary, there are two special frames of reference we might choose for convenience during our calculations. Whichever frame we choose we should get the same result. The frames are:

[4] In collisions between springy/deformable objects, the peak pressure is also lower, because the force tends to be distributed over a larger surface area.

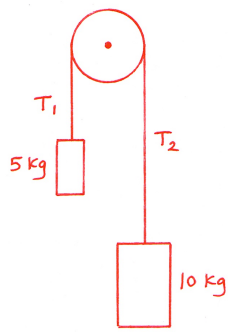
- A frame in which one body is stationary, allowing us to simplify the equations.
- The *centre-of-momentum frame*, in which momentum is zero both before and after the collision (by the law of conservation of momentum). This frame has the special property that in an elastic collision the bodies rebound with the same speeds as before the collision (but with the signs of the velocities reversed).

4.1 Pulleys ★

Many students find this straightforward question (in the sense that there are no tricks) harder than they expect.

A LIGHT, INEXTENSIBLE rope connecting 5 kg and 10 kg masses is suspended over a light frictionless pulley. The system is stationary, and has just been released from rest at the instant shown. Answer the following questions:

1. Are the tensions T_1 and T_2 the same?
2. What is the magnitude and direction of the force on the pulley?



Answer

Let us consider the two parts of the problem in turn.

1) ARE THE TENSIONS T_1 AND T_2 THE SAME?

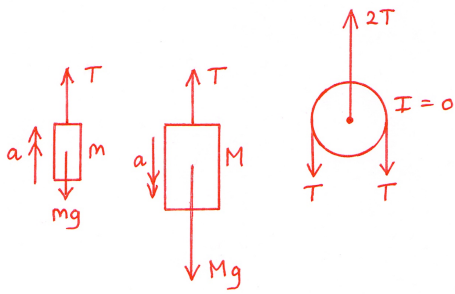
The simple answer is yes, they are the same. They have to be the same because both the rope and the pulley are *light*, so have no inertia.[5] Likewise, the rope is *inextensible* so has no ability to store energy. If we pull on one end, the force must be transmitted to the other end.

[5]Here we can think of *mass* as being the *translational inertia*. The *rotational inertia* of the pulley is defined by $I = \sum mr^2$, the sum of the mass elements multiplied by their distance from the centre of rotation.

1) What is the Magnitude and Direction of the Force on the Pulley?

For this part we need to solve the forces on all the objects. To do this we need to draw free-body diagrams of all three components. To simplify the diagram we let the smaller weight have mass m, and the larger weight have mass M. We have established that the tensions in the ropes are equal. We represent the tensions by T.

Let's deal first with the pulley. It should be immediately apparent that to achieve zero acceleration of the pulley, we require an upward force equal to $2T$ acting from the axle of the pulley. We now need to establish the value of T.



Consider the smaller mass m. Resolving forces *upwards positive*, and allowing the mass to have a *positive upward* acceleration a, we have

$$T - mg = ma$$

Consider the larger mass M. Resolving forces *downwards positive*, and allowing the mass to have a *positive downward* acceleration a, we have

$$Mg - T = Ma$$

Adding these equations we have

$$(M - m)\, g = (M + m)\, a$$

Rearranging for a we have

$$a = \frac{(M - m)}{(M + m)} g$$

Putting in values ($m = 5$ kg, $M = 10$ kg) we have $a = (1/3)\, g$. We can now solve either of the equations for the masses to find T. Doing this gives $T = (20/3)\, g$. The upward force on the pulley is equal to $2T$, so is therefore $(40/3)\, g$. Taking $g \approx 10$ m s^{-2} gives a force of approximately 133 N.

We can compare this value to the upward force on the pulley that would be required if the system were prevented from rotating (and the rope were

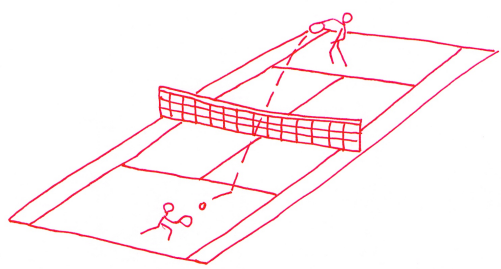
prevented from slipping over the pulley). In this case we would need to support the entire weight of the masses, and the upward force would be $15g$, or approximately 150 N. By allowing the centre of mass of the system (which comprises both masses and the pulley[6]) to accelerate downwards, we reduce the force requirement on the pulley axle.

4.2 Dr Lightspeed's elastotennis match ★★

Dr Lightspeed and Professor Bumble are lining up for another elastotennis match. Viewed from a distance it looks an awful lot like ordinary tennis with very good players. The secret to their apparent prowess is that the elastoball and elastoracket are both perfectly elastic. It is a futile ceremony in which Dr Lightspeed wins every point. Fortunately, the ritual usually only lasts a couple of sets before the cucumber sandwiches arrive and the pair tackle some problem in mathematics.

"It is fascinating," says Professor Bumble, "how pre-eminently fast your returns are. I swear the ball comes off your racket faster than the sum of the ball speed and the swing speed. That should be impossible." Dr Lightspeed looks confused for a moment then replies, "Why, dear professor, surely the reverse is true. In elastotennis it would be impossible for it *not* to come off my racket faster than the sum of those speeds." They return to the match, in which Dr Lightspeed wins every point.

Later, over more sandwiches, the professor is rather subdued. They sit nibbling in silence. Eventually the professor speaks. "I still don't understand, Dr Lightspeed, how you can return so fast. You really must explain."



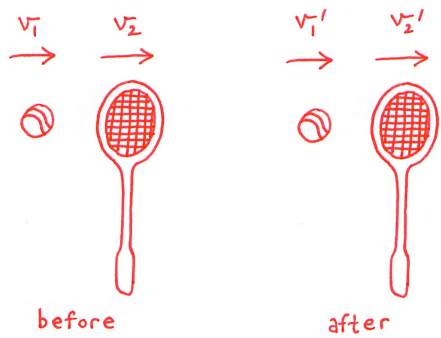
In elastotennis (tennis in which both the ball and the racket are entirely elastic), what is the maximum return speed of the ball in terms of the original ball speed and swing (racket) speed?

[6]Though note that, in this case, the pulley has no moment of inertia.

Answer

First we will take a brute-force approach to the problem, the standard method. We'll solve the momentum and energy equations in the global frame of reference (the frame of reference of the tennis court). I've also given an alternative answer, which exploits the centre-of-momentum frame, and which is more elegant.

In the global frame of reference we define the velocities of the elastoball and the elastoracket as v_1 and v_2 respectively.



We arbitrarily define a positive value of velocity as corresponding to motion to the right, and a negative value of velocity as motion to the left. Letting the masses of the ball and racket be m_1 and m_2, we can write the momentum equation

$$m_1 v_1 + m_2 v_2 \quad = \quad m_1 v_1' + m_2 v_2'$$

or

$$v_1 + r v_2 = v_1' + r v_2'$$

where $r = m_2/m_1$. For a completely elastic collision we have

$$e = -\left(\frac{|\boldsymbol{v}_1' - \boldsymbol{v}_2'|}{|\boldsymbol{v}_1 - \boldsymbol{v}_2|}\right) = 1 \quad \text{or} \quad v_1 - v_2 = v_2' - v_1'$$

where v_1' and v_2' are the velocities after the collision. Eliminating v_2' we have

$$v_1' = \frac{v_1\,(1-r) + 2 r v_2}{(1+r)}$$

Taking v_1 to be positive (the ball moving to the right) and v_2 as negative (the racket moving to the left), we seek to make v_1' as negative as possible. We therefore look at both limits, $\lim_{r\to 0}(v_1')$ and $\lim_{r\to\infty}(v_1')$, and investigate all possible minima. Let's consider each of the three in turn.

- CONSIDER $\lim_{r\to 0}(v_1')$. We see that $\lim_{r\to 0}(v_1') = v_1$. This is because m_1 is so heavy (relative to m_2) that it continues unimpeded after the collision.

- CONSIDER $\lim_{r\to \infty}(v_1')$. We see that $\lim_{r\to \infty}(v_1') = 2v_2 - v_1 < 0$. In this case m_2 is so heavy (relative to m_1) that it continues unimpeded after colliding.[7] The smaller mass m_1 (the ball) bounces off the unaffected larger mass m_2 (the racket).

- CONSIDER ALL POSSIBLE MINIMA. To do this, we first differentiate to get dv_1'/dr, giving

$$\frac{dv_1'}{dr} = \frac{2\,(v_2 - v_1)}{(1+r)^2}$$

  However, $v_1 > 0$ and $v_2 < 0$, so $v_2 - v_1 < 0$. For finite m_1 and m_2, $r = \infty$ is not allowed. For finite r, $dv_1'/dr \neq 0$, so there is no minimum or maximum.

Considering both limits, and having shown that there are no minima or maxima, we see that $\lim_{r\to \infty}(v_1') = 2v_2 - v_1$ is the solution. Recalling that $v_1 > 0$ and $v_2 < 0$ we see that the maximum speed of return $|v_1'|$ is equal to the ball speed plus twice the racket speed. We write

$$|v_1'| = 2\,|v_2| + |v_1|$$

Dr Lightspeed is correct. In elastotennis it is impossible for the elastoball *not* to come off the racket faster than the sum of the ball speed and the racket speed.[8] No wonder Professor Bumble is struggling to return his shots.

An elegant approach

We now consider an approach that invokes the centre-of-momentum frame of reference.

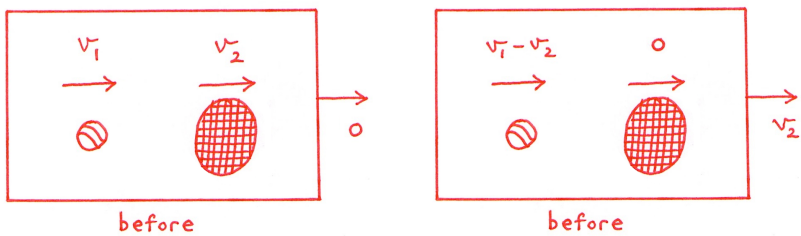
We are interested in determining the *maximum* return speed. This will occur when the change in velocity of the ball is greatest, no matter what inertial frame of reference we consider the problem in. Consider the frame of reference in which the elastoracket is stationary. It *should* be obvious that in an elastic collision, the biggest change in velocity of the ball occurs when the change in velocity of the racket during the collision approaches zero. This condition occurs when the racket has a much greater mass than the ball ($m_2 \gg m_1$). If this doesn't feel right, even in the frame of reference of the

[7] To show this, we consider the symmetry of the problem (that is, what would happen if we interchanged 1 for 2 throughout the analysis). Alternatively we can solve directly for v_2'.

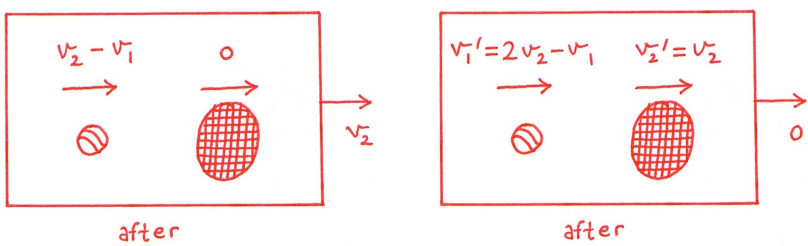
[8] For an individual shot this is true even for realistic ball and racket weights (see further discussion).

racket, try and invent some examples to convince yourself.[9] In the discussion we solve the more general problem (m_1 and m_2 having the same order of magnitude[10]).

Because we can perform our calculations in any inertial frame of reference, and because for $m_2 \gg m_1$ the racket will experience no velocity change during the collision, we consider the problem in the frame of reference of the racket. This frame is travelling at v_2. The velocities v_1 and v_2 in the global frame of reference become $v_1 - v_2$ and 0, respectively, in the racket frame. We represent the velocities in both frames below, and show the velocities of the frames with respect to the global frame.



Considering the collision in the racket frame, for $e = 1$, and for $m_2 \gg m_1$, the velocity of the ball is reversed. A ball velocity before collision, $v_1 - v_2$, becomes a velocity after collision, $v_2 - v_1$. Because $m_2 \gg m_1$, the velocity of the racket after the collision is still zero. We now step back into the global frame by adding the racket frame velocity, v_2, to the velocities of the ball and racket within the racket frame. For $m_2 \gg m_1$, the velocities of the ball and racket after collision are $v_1' = 2v_2 - v_1$ and $v_2' = v_2$.



[9]Imagine floating in a space station with a superball (a highly elastic toy ball) and a number of hard steel plates of various thicknesses. We throw the superball at a particular floating stationary plate. If the plate is very heavy, the ball rebounds with approximately the same speed as it had when it arrived. This is like bouncing the ball off a hard floor. As we make the plate lighter, the ball rebounds more slowly. When the plate is very much lighter than the ball, it can no longer impede the ball's progress, and the ball does not return.

[10]The *order of magnitude* of a number is its size to the nearest power of 10, and is used to make approximate comparisons of the sizes of two numbers. The order of maguitude of a number a is $\log_{10}(a)$ rounded to the nearest integer. If two numbers a and b have the same order of maguitude we write $a \sim b$.

We see that by using this approach we achieve the same result with considerably less complexity. The centre-of-momentum frame can be a powerful tool.

A realistic case ★★★

Believe it or not there are quite a number of academic papers on the physics of tennis. Specifically ball-racket interactions, and ball-surface interactions.[11] In both cases the coefficients of restitution are reasonably similar, and both are functions of the impact speed. The latter defines the deformation of the tennis ball, which is the main source of energy loss during the collision. The coefficients of restitution vary between about $e = 0.7$ at low speed (10 m s^{-1}), and about $e = 0.4$ at high speed (45 m s^{-1}).

Let's reconsider the problem above, but with realistic values everywhere. A fast approach speed for a ball after bouncing is $v_1 = 20$ m s^{-1}. A very fast forehand racket speed would be $v_2 = -40$ m s^{-1}. A typical ball weighs about $m_1 = 60$ g, and a typical racket weighs about $m_2 = 300$ g. Assume that the coefficient of restitution is $e = 1/2$.

We step into the centre-of-momentum frame of reference, defined as moving with velocity V relative to the court frame. The velocities of the ball and racket relative to the centre-of-momentum frame are $u_1 = v_1 - V$ and $u_2 = v_2 - V$. In this frame, by definition, the momentum is zero. That is, $m_1 u_1 + m_2 u_2 = 0$. Expanding this we find that

$$m_1 (v_1 - V) + m_2 (v_2 - V) = 0$$

or

$$V = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = -30 \text{ m s}^{-1}$$

This gives $u_1 = 50$ m s^{-1} and $u_2 = -10$ m s^{-1}. In the centre-of-momentum frame the relative speeds of the ball and racket are $u_2 - u_1 = -60$ m s^{-1}. The coefficient of restitution is $e = 1/2$. We denote the speeds of the ball and the racket in the centre-of-momentum frame after collision as u_1' and u_2'. The relative speeds after collision are therefore given by $u_2' - u_1' = -0.5\,(u_2 - u_1) = 30$ m s^{-1}. In the centre-of-momentum frame we must have zero momentum after the collision. That is, $m_2 u_2' + m_1 u_1' = 0$. Satisfying both of these conditions yields $u_1' = -25$ m s^{-1} and $u_2' = 5$ m s^{-1}. To step out of the centre-of-momentum frame and into the court or global frame, we need to add the frame velocity. The velocities of the ball and racket in the court frame are therefore $v_1' = u_1' + V = -55$ m s^{-1} and $v_2' = u_2' + V = -25$ m s^{-1}.

In this more realistic case we achieve a ball velocity of 55 m s^{-1} (123 mph), for racket and ball speeds of 40 m s^{-1} and 20 m s^{-1} respectively. For a per-

[11] For further discussion see: Miller, S., 2006, "Modern tennis rackets, balls, and surfaces," British Journal of Sports Medicine, May 2006, 40(5), pp. 401–405, doi: 10.1136/bjsm.2005.023283; and Brody, H., 1997, "The physics of tennis—III—The ball–racket interaction," American Journal of Physics, 65, pp. 981–987.

Chapter 9

Electricity

In this chapter we look at problems involving *electricity*. Some of the problems test the application of very simple concepts, such as Ohm's Law, or the definition of electrical power. You may feel that some of the questions are too simple even for high-school students. They are not, however, and it is surprising how many students struggle to think their way clearly to a cogent answer. Problems like this can be used to determine whether students have a sound knowledge of GCSE-level material on these topics. We must not dismiss them.

I have also included quite a few *resistor puzzles*, which involve groups of resistors either in two-dimensional or three-dimensional arrays. The goal of these is to calculate the overall resistance of the network. Resistor puzzles have been very popular over the years. The classic question is that of the resistor cube, where each edge of the cube has a given resistance (R, say) and the aim is to calculate the overall resistance between two vertices, most commonly the longest diagonal (this is also the simplest variant). The rules for solving these problems are simple, so they reduce almost to logic problems.

I have not included any *unsteady problems*—those which involve the charging and discharging of capacitors, or the time constant of CR circuits, etc.—even though these are also quite popular pre-university puzzles. They are easy to find in most textbooks, and are rather formulaic, so (to me anyway) are therefore less interesting. Nonetheless, this is an important area that students should be familiar with if they've studied physics to pre-university level.

Let's review a few simple definitions that will help us with these problems.

- Current. The current, I, is related to the charge, Q, according to the definition $I = dQ/dt$, where t is time. For steady currents this reduces to $I = Q/t$.

- Ohm's Law. The current, I, through a conductor is proportional to the potential difference, V, across it, and inversely proportional to the

resistance[1] R. We write $I = V/R$.

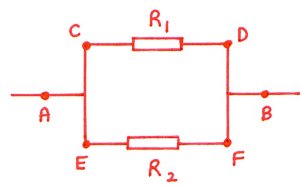
- Electrical Power. The electrical power, P, dissipated in a load of resistance R, is proportional to both the current through the load, I, and the potential difference across it, V. We write $P = IV$. Combining this with Ohm's Law we can write instead $P = I^2R$ or $P = V^2/R$.

- Resistors in Series. Resistors in series add. In other words, the total resistance is given by $R_T = R_1 + R_2$.

We should prove this relation, because it is used in many of the problems that follow. It is a consequence of Ohm's Law: $V = IR$. The *voltage drops* across R_1 and R_2 happen in series, so the total voltage drop between A and C is given by $V_T = V_1 + V_2$, where V_1 and V_2 are the individual voltage drops. The current flowing from A to B is the same as the current flowing from B to C,[2] so we can rewrite this equation as $IR_T = IR_1 + IR_2$. Dividing the equation by I, we get $R_T = R_1 + R_2$.

- Resistors in Parallel. Resistors in parallel obey the following rule.

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$$



To prove this we note that for a given voltage drop V_{AB} the voltage drop between C and D must be equal[3] to the voltage drop between E and F. That is, $V_{CD} = V_{EF} = V_{AB}$, or $V_1 = V_2 = V$. The current, on the

[1] This is only true when the resistance is *not* a function of the current—that is, $R \neq f(I)$. For many real loads in practical situations this assumption may not be valid.

[2] This is because charge is not leaving the circuit, and because there is no place for charge to accumulate. We can think of this as a one-dimensional continuity equation for a conserved quantity (charge).

[3] Taking the wire to have zero resistance, and noting that voltages add in series, we have $V_{CD} = V_{CE} + V_{EF} + V_{FD}$. The resistances between C and E and between F and D are both zero, giving $V_{CE} = 0$ and $V_{FD} = 0$. So $V_{CD} = V_{EF}$.

other hand, obeys an additive law: $I_T = I_1 + I_2$. Combining this with Ohm's Law we get

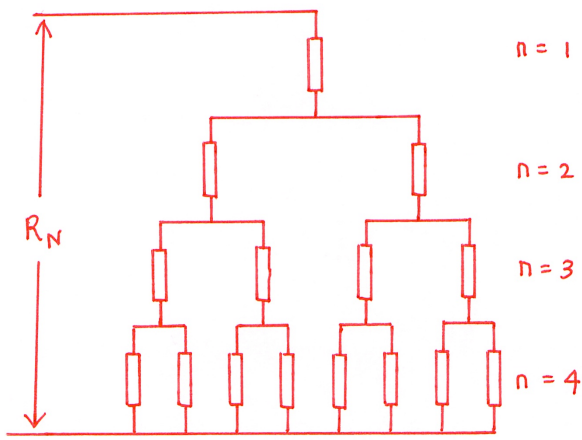
$$\frac{V}{R_T} = \frac{V}{R_1} + \frac{V}{R_2} \text{ or } \frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$$

- Capacitors in Series. Capacitors in series obey $1/C_T = 1/C_1 + 1/C_2$.
- Capacitors in Parallel. Capacitors in parallel obey $C_T = C_1 + C_2$.
- Energy Stored in a Capacitor. The energy stored in a capacitor is given by $E = (1/2)\,CV^2$.

We are now ready to try a few puzzles. As I remarked, the resistor cube is a well-established puzzle that has been circulating for several decades—I've also seen it in at least one physics textbook. The other resistor questions are my own, the result of a productive hour spent at Detroit airport waiting for a connecting flight across the US. Every time the waitress topped up my coffee she asked if I had solved my problem yet. "Yes," I said, "but I keep finding new problems." To which she replied, "Oh dear, honey, you just keep at it then." Good advice for anyone attempting these puzzles.

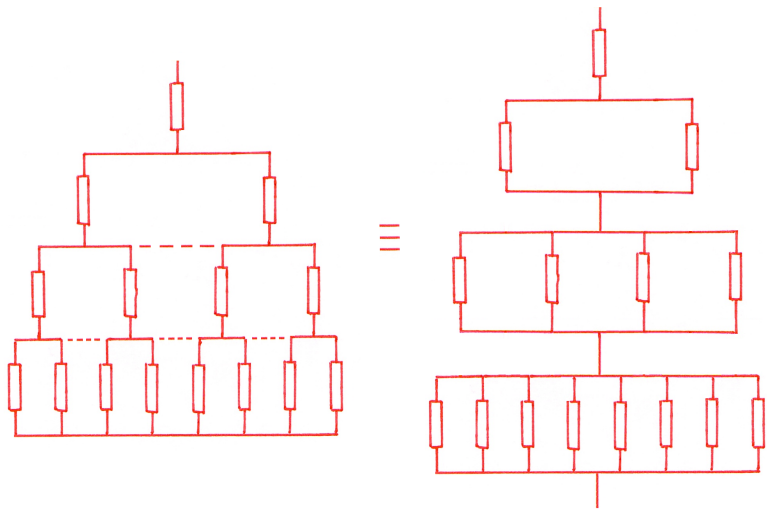
9.1 Resistor pyramid ★★

A resistor pyramid is built in a branched structure as shown below. If there are N levels, constructed from individual resistances of value R, what is the total resistance R_N across the pyramid? What is the resistance of an infinite pyramid of this type?



Answer

We can think of the structure of the pyramid as a combination of both series and parallel connections of resistors. From the symmetry of the arrangement we see that all nodes at a given level n in the pyramid are *equipotential points*. We can add *dummy wires* (shown as dotted lines in the diagram) between these equipotential points without changing the current flows in the circuit. So we see that the equivalent circuit has n levels in series, with 2^{n-1} resistors in parallel at the n-th level.



The resistance of the n-th level is therefore

$$R_n = \frac{R}{2^{n-1}}$$

The resistance of a pyramid of size N is the sum of the resistances R_n from $n = 1$ to N. That is,

$$R_N = \sum_{n=1}^{N} \frac{R}{2^{n-1}} = R\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{N-1}}\right)$$

The right-hand side of this expression is a geometric series[4] with the sum[5]

$$R_N = 2R\left[1 - \left(\frac{1}{2}\right)^N\right]$$

[4] A geometric series of n terms has the general form $S_n = a(1 + r + r^2 + \ldots + r^{n-1})$, where a is the value of the first term and the common ratio is r. The sum can be expressed simply as follows. We multiply by r to give $rS_n = a(r + r^2 + r^3 \ldots + r^n)$, then subtract this from the previous expression to give $rS_n - S_n = ar^n - a$. Rearranging, $S_n = a(r^n - 1)/(r - 1)$.

[5] Taking $a = R$ and $r = 1/2$.

To calculate the resistance of an infinite pyramid we need the sum to infinity of the series. The common ratio is $r = 1/2 < 1$, so the sum to infinity converges to give

$$R_\infty = 2R$$

It might have been easier to put two resistors in series.

A current-symmetry argument

Another way of thinking about this problem is in terms of current symmetry in each tier of the pyramid. At the n-th level of the pyramid there are 2^{n-1} resistors, each with the same current flowing through them (due to the symmetry of the structure). If the current through the entire pyramid is I, then the current through each resistor in the n-th level is $I_n = I/2^{n-1}$. We need to add the potential (or voltage) drop across each tier in series to find the potential drop across the entire pyramid. The potential drop across a resistor in the n-th level is $V_n = I_n R = IR/2^{n-1}$. So the total potential drop across a pyramid of N levels is

$$V_N = \sum_{n=1}^{N} \frac{IR}{2^{n-1}} = IR\left(1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{N-1}}\right) = 2IR\left[1 - \left(\frac{1}{2}\right)^N\right]$$

The resistance of a pyramid of N levels is given by $R_N = V_N/I$. As before, we get

$$R_N = 2R\left[1 - \left(\frac{1}{2}\right)^N\right] \text{ and } R_\infty = 2R$$

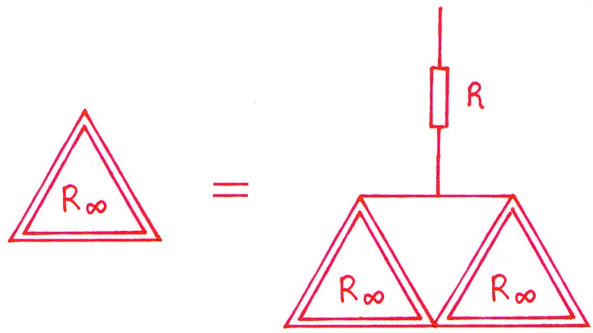
An elegant solution

We now consider an elegant solution that will appeal to the more mathematically minded. A friend came up with the idea, and my brother and I extended it.

In the diagram we represent an infinite pyramid of resistors as a double triangle inscribed with R_∞. If we extend the pyramid by one level, it's the same as having two such pyramids in parallel, in series with a single resistor. If the series converges, the resistance of the extended pyramid must be the same as that of the original pyramid. We write

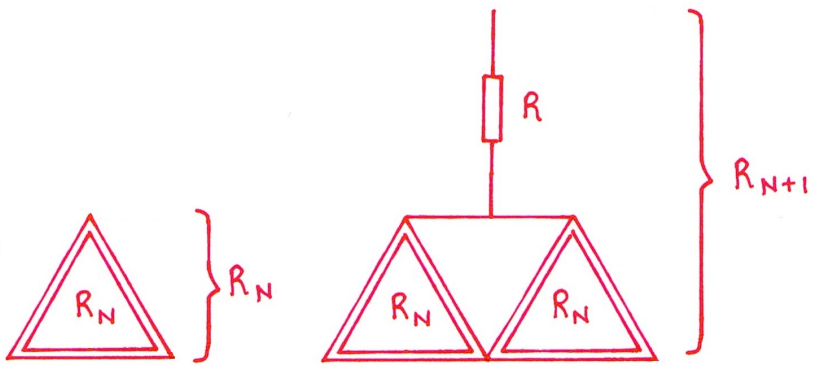
$$R_\infty = R + \frac{R_\infty}{2}, \quad \text{so} \quad R_\infty = 2R$$

A very elegant solution.



Let's consider how we might extend this to find the resistance of a pyramid with N levels. In the diagram we represent a pyramid with N levels as a double triangle inscribed with R_N. We see that extending the pyramid by one resistor gives

$$R_{N+1} = R + \frac{R_N}{2}$$



Here we remind ourselves that R_N and R_{N+1} are terms in a series that we seek the expression for. Noting that the sum to infinity of the series is given by $R_\infty = 2R$, we define a new series $Q_N \equiv R_N - 2R$. The series Q_N converges on zero—that is, $Q_\infty = 0$. We substitute the definition of Q_N into the expression for R_{N+1}, giving

$$Q_{N+1} + 2R = R + \frac{(Q_N + 2R)}{2}$$

So

$$Q_{N+1} = \frac{Q_N}{2}$$

Noting that $R_1 = R$, and that therefore $Q_1 = -R$, we see that

$$Q_N = \frac{-R}{2^{N-1}} = -\frac{2R}{2^N}$$

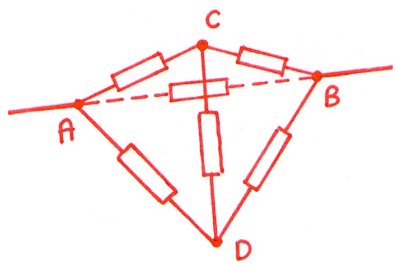
Using $Q_N \equiv R_N - 2R$, we get the expression for R_N.

$$R_N = -\frac{2R}{2^N} + 2R = 2R\left[1 - \left(\frac{1}{2}\right)^N\right]$$

If the previous solution was elegant, this is elegance to the power N.

9.2 Resistor tetrahedron ★

A RESISTOR TETRAHEDRON has individual resistance values R in each leg. What is the total resistance R_T across any pair of nodes in the pyramid?



Answer

It is easiest to think about this by flattening the structure onto a plane. Doing this makes it easier to see the essential structure we're dealing with. Let's consider the total resistance between A and B. When it's flattened—as shown in the diagram—we can see that the network reduces to a single resistor (at the top, $A \to B$) in parallel with a group of five resistors (at the bottom, $A' \to B'$). If all the individual resistances take the same value R, the resistance of the bottom group itself becomes R. We see this by noting from the symmetry of the arrangement that the potentials at C and D are the same, so no current flows between those two points. Thus the resistor between C and D can be removed without affecting the currents in the circuit. The circuit is then equivalent to having two resistors of value $2R$ in parallel, giving a resistance of R for this block. The resistance of the top resistor (between A and B) is trivially R. The total resistance between A and B is therefore $R_T = R/2$. By symmetry, this is the resistance between any two nodes.

Chapter 11

Optics

In this section on *optics* we look at problems built on simple principles of reflection and refraction. The underlying material is no more than GCSE level in difficulty, but the questions are designed to be challenging and unusual. Some are well-known questions, and others are my own invention.

- Law of Reflection. The angle between the incident ray and the normal is the same as the angle between the reflected ray and the normal.

- Snell's Law of Refraction. A ray of light passing through an interface between two different media of refractive indices n_1 and n_2 obeys Snell's Law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where θ_1 and θ_2 are the angles to the normal of the incident and refracted rays.

11.1 Mote in a sphere ★

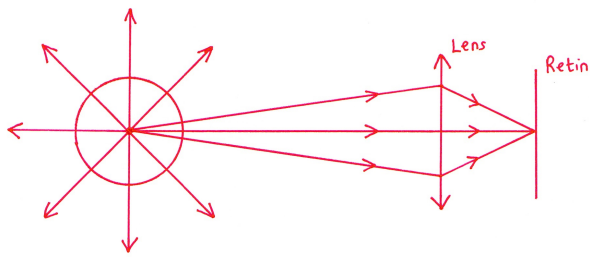
I have only ever heard the word *mote* used in two places. The first was the Bible,[1] in which one appears in a person's eye. The second was in my Oxford physics interview, during which I had to answer the following optics puzzle. Fortunately, at my school a passing knowledge of the Bible was inescapable, so I knew that a mote was simply a small speck of something—a dust particle, for example. However, I'd not seen any unusual optics questions before then—but I suppose that was the point. The question is as follows.

A mote is at the very centre of a perfect sphere of glass. Where, if anywhere, do you see the mote?

[1] It appears in the Sermon on the Mount, as a moral lesson against hypocrisy and self-righteousness. The word *mote* comes from Greek, and means a very small dry body. Today we use the word *speck* more commonly.

Answer

We are, of course, expected to draw a ray diagram to demonstrate the solution. Snell's Law tells us that refraction occurs according to the rule $n_1 \sin\theta_1 = n_2 \sin\theta_2$. Because the mote is at the very centre of the sphere, all emitted rays have an angle of incidence of $\theta_1 = 90^\circ$ with the glass-air interface. So the rays are not diffracted by passing through the interface, and the mote appears to be exactly where it is. In the ray diagram we represent the eye with a converging lens, showing it focussing the mote down to a point on the retina.

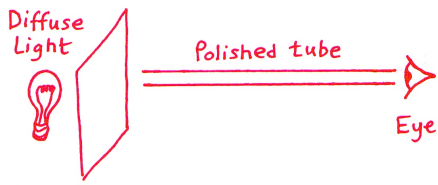


11.2 Diminishing rings of light ★★★

I first heard this question about 15 years ago, though I think it is much older than that. My PhD supervisor was also a great fan of physics puzzles, and this was one of his favourites. I used to think the question was too hard for high-school students, but tried it on a few recently and found that with enough hints students made surprisingly good progress, and most were able to complete the question with a bit of help. I have also tried it on a number of PhD students, who did no better. I attributed this to them forgetting most of their high-school physics: they were all engineers. It is a very enjoyable question to ask, and best done with an experiment. First I would ask students to describe what they saw before deriving equations from first principles to explain it. Students really enjoy this, and find it very satisfying when they manage to derive equations for the interesting optical display. When you attempt the question, remember that it works best with an experiment and someone to give you hints.

YOU ARE GIVEN a long, thin polished metal tube (a 1 m length of the copper

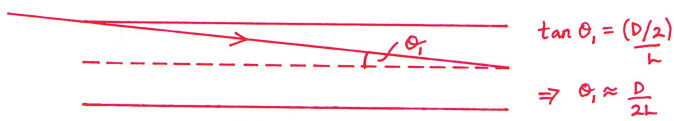
pipe used in domestic plumbing, for example) and asked to put your eye at one end and to point the other end at a diffuse light source (a well-illuminated wall, for example). What you see is a pattern of rings. At the centre there is a bright circle, which is the diffuse light source seen directly through the hole in the end of the tube. Surrounding this is a ring of light that is less bright than the central circle. Around that there's another ring that's less bright still. And so on. The rings become successively less bright until the pipe appears dark. The rings appear to have approximately the same width as they get larger. You are asked to derive an equation for the angle subtended at the eye by the n-th ring.



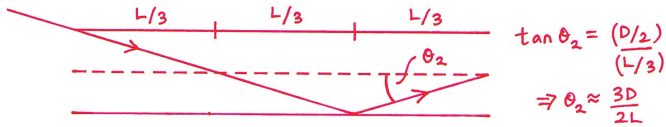
Answer

When approaching this question, it may help to first sketch a diagram of the tube, with the centre marked as a dotted line, in preparation for drawing ray diagrams. Analyse first the light that is directly incident at the eye from the diffuse light source without reflection, then consider the first-order reflection, second-order reflection, and so on.

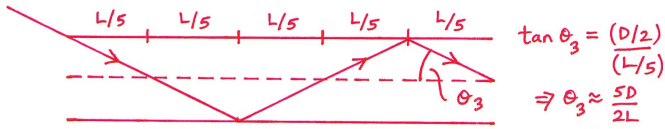
We start by drawing the ray that subtends the highest angle at the eye (which is at the centre of the tube). We see that where the diameter of the tube is D, and where the length of the tube is L, the highest angle that can be subtended at the eye without reflection is given by $\tan\theta_1 = (D/2)/L$. For a long tube, taking the *small-angle approximation*, we have $\theta \approx D/2L$.



We then consider the first reflected ray. To do this, the only additional thing we need to know is the law of reflection, which is simply that the incident and reflected rays make the same angle with the wall. We see that the ray that subtends the highest angle at the eye, and which is reflected once, divides the tube into three. We have $\tan\theta_2 = (D/2)/(L/3)$. That is, $\theta_2 \approx 3D/2L$.



We then consider the second reflected ray. The ray that subtends the highest angle at the eye, and which is reflected twice, divides the tube into five. We have $\tan\theta_3 = (D/2)\,/\,(L/5)$. That is, $\theta_3 \approx 5D/2L$.



We quickly see that the general formula for the angle subtended at the eye by the n-th ring is

$$\tan\theta_n = \frac{(D/2)}{(L/\,(2n-1))}$$

Taking the small-angle approximation, we see that

$$\theta_n \approx \frac{(2n-1)\,D}{2L}$$

This is a rather satisfying question, and one that works very well with the accompanying demonstration. If you haven't already done so, track down a length of copper pipe a meter or so long (from your local plumber's shop if necessary) and try it. It is very neat indeed.

11.3 Floating pigs ★★★

If you've read the introduction to this book, you'll know that I was asked this in my university physics interview. It's difficult to recreate here the manner in which the question was asked, because it was set up as a desktop experiment. I was asked to explain the strange optical effect that it created. That is certainly the most logical way to ask the question. I still vividly remember the demonstration. The interviewer took two bowls from a drawer, dusted them off with a handkerchief, and placed them proudly—and with something of a flourish—on the table in front of me. They were about the size of cereal bowls, and their insides were perfect mirrors. The top bowl had a hole about two inches in diameter in its base, and was set upside-down on top of the lower bowl, creating a mirrored cavity which could be viewed through the hole. Before that, though, the interviewer placed a small, bright pink plastic pig—about the size

Chapter 14

Estimation

I really enjoy questions on *estimation*, or back-of-the-envelope calculations. They are popular questions for a whole range of numerate disciplines, and are certainly very common elements in physics and engineering tests. Strangely, they are also very common in job interviews, particularly with companies that consider themselves to be "free thinking".

I remember several summer evenings when we were physics undergraduates, sitting in a friend's room, staring out over the cricket lawn, drinking coffee, and estimating things simply for the *fun* of it. The game had two stages. First, we would *guess* the answer to the problem at hand, going on instinct alone, and then compare our guesses. At this stage, to raise the stakes, you had to pretend to be very confident about your own guess and very sceptical about everyone else's. It didn't matter that we all made the least accurate guess about the same number of times. Then we would independently *estimate* the answer, usually without recourse to pen and paper. It could take a while, because we often had to devise strategies to do the calculation at hand. This was just before the internet changed everything. So if we needed some important but esoteric value we couldn't estimate easily, one of us would be sent to the library to consult a (printed) encyclopaedia, or to look at a set of data tables, and would reappear with the population of Bolivia, say, or the density of uranium, scribbled on a scrap of paper. Once we'd made our estimates we would compare them. It was often surprising how close the estimates were, even when we had taken diverging approaches to the solution.

We worked out that there were fewer blades of grass on the Merton College cricket ground than there were people in the world,[1] and that if we shared the books in our college library equally across the population of the United

[1] Merton College cricket ground is about 200 yards on a side, and we estimated about 50 blades of grass per square inch. This gave $50 \times 12^2 \times 3^2 \times 200^2 = 2.6 \times 10^9$ blades. Back in the mid-1990s this was roughly half the population of the world, which was 5.7 bn (it is now over 7 bn). I still remember gazing out over the brown field at the end of that summer and thinking what an *awful* lot of people that represented, and how insignificant we would each seem if we all stood there side by side.

Kingdom (something which we felt like doing more than once) we could offer each citizen a paltry third of a page.[2] We worked out how many college rooms it would take to store all the gold in the world—a question that appears in this book.

Wherever you look there is something to estimate, and it's useful to know how to set about it. Just recently someone told me that work had started on the Kingdom Tower in Saudi Arabia. If it's completed it will be over a kilometre high. The energy cost of pumping water up the tower, she speculated, will be terrific. I thought for a moment, and with a little mental arithmetic worked out the annual cost per person. When I told her the number she looked at me in surprise. "How did you know that?" she asked. When I told her I had just estimated it, but fairly accurately, she looked at me in disbelief. I had to go through the steps of the calculation to convince her that I really had just calculated it in my head.

Most everyday things can be estimated easily and relatively accurately. It's an essential skill for anyone who is serious about being a scientist. I shouldn't really admit to it here, but I'm always a little shocked when someone who has been through high-school science can't estimate—roughly—how big, long, heavy etc. some object of interest is. Often people simply shrug their shoulders and say, "I have no idea." But we should always have *some* idea.

I hope you will enjoy this selection of problems. They are based on the more unusual questions I have heard over the years, or are favourites of my own. Some of them require the odd value which you might not be able to estimate accurately enough to satisfy yourself. In which case you might need to look it up in a data table, or ask your tutor for it. Not knowing values is fine. It's the ability to come up with a *method* for solving a problem that's important.

14.1 Mile-high tower ★

Frank Lloyd Wright, the great American architect, was perhaps not the first to conceive of the notion of a mile-high building, but he was probably the first to work seriously on plans for one. His 1956 skyscraper "Illinois" would have been 528 stories high, making it four times taller than the Empire State Building. To get up and down it you'd have had to use one of 76 "atomic-powered" elevators (gulp!). The Illinois was never built, and it's unlikely that the technology to do so even existed at that time.

Almost sixty years later, the tallest structure ever constructed by man is the Burj Khalifa in Dubai, in the United Arab Emirates. Finished in 2010, it stands 829.8 m tall. Huge as it is, it is still only half the height of the magic mile.

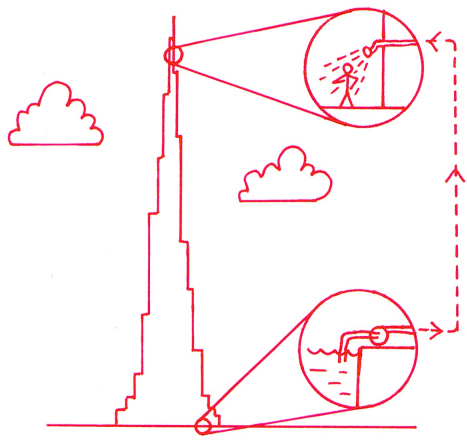
[2]The library walls totalled a good 300 m, and were stacked six shelves high. A *ream* of 500 pages of paper is about 5 cm thick. This gives $(300 \times 6 \times 500)/0.05 = 18 \times 10^6$ pages, or a third of a page for each of the then 58 million inhabitants of the UK (the population is now over 63 million).

The Kingdom Tower, which is currently being built in Jeddah, Saudi Arabia, was originally designed to be one mile high. Plans were drawn up by the architect who designed the Burj Khalifa. But the geology of the area proved unsuitable, and the tower had to be scaled down. The current intended height is still a closely guarded secret, but it has been announced that it will be over 1 km, setting a new record for super-tall buildings.

In the architectural world there now seems to be little debate over the technical feasibility of the mile-high building—the question is purely one of finance. Given the sudden explosion in super-tall buildings in the last ten years, the near future will be very interesting.

I remarked in the introduction that someone I know recently speculated that the cost of pumping water to the top of a mile-high building would be fantastic. The purpose of this question is to calculate whether or not that is so.

ESTIMATE THE ANNUAL financial cost of the energy required per person to supply water to living accommodation at the very top of a mile-high building.



Answer

Of course, we have to assume that we're talking about the current economic and energy climate. The current cost of electricity in the UK is roughly 15 pence per kWh. It's reasonable to expect students to be able to estimate this to within, say, a factor of two. Energy is a key household expense, and one of the most significant drivers of the economy. Now let's calculate the annual energy required to pump water for a single person.

First we need to calculate the amount of water used by a person in a day. We can create a rough table to show this, and each of us might fill the table slightly differently. But we should certainly agree to within a factor of two or three, say. My rough estimates are as follows.

	Number per day	Litres	Total litres
Toilet flushes	5	10	50
Showers	2	25	50
Washing clothes	1/3	60	20
Washing dishes	1	20	20
Sundries	1	10	10
		Total	**150**

We estimate the total amount of water used at 150 l (or 150 kg) per day.

Now we can estimate the energy required to pump that water to the top of a mile-high tower. A mile is approximately 1.6 km, or $1,600$ m. The energy cost is simply the gravitational potential energy gained, mgh, divided by the total efficiency of the pump. Let's take a very conservative electrical efficiency value of $\eta_E = 0.5$, and a very conservative mechanical efficiency value of $\eta_M = 0.5$. The total energy cost per day is

$$E = \frac{mgh}{\eta_E \eta_M}$$

Taking $m = 150$ kg, $g \approx 10$ m s^{-2} and $h = 1,600$ m gives $E = 9.6 \times 10^6$ J. This looks like a big number, but let's put it in context by calculating the number of kWh it represents. One kWh is equal to 3.6×10^6 J, so we need 2.7 kWh to pump a day's water to the top of a mile-high tower. That works out at a daily cost of approximately 40 pence. The annual cost of the energy is approximately £146. It's not a trivial sum, but it is far from being a fantastic amount of money. In fact it's only about one-third of the cost of a typical Oxford water bill. In the scheme of things we can say it is fairly insignificant.

The Burj Khalifa in Dubai has set an unprecedented record for height, and it's likely that it will soon be surpassed by the Kingdom Tower in Jeddah, at over 1 km high. The smart money says that in the next few years we will build even higher. Consider the predictions made in a recent paper in the Journal of Urban Technology:[3]

> Technologically, a one-mile-high (1.6 km) building is possible today—although not without the associated challenges of required high-strength materials, fire safety provisions, elevator systems, and, above all, constructability. As the building gets taller, the slenderness (height-to-width) ratio increases and the tower becomes dynamically extra sensitive to wind so it cannot be inhabited comfortably. To overcome this, the slenderness effect of the tower is offset by making it wide at the base, or connecting several towers with bridges at different levels so they will enforce and stiffen each other. Will it be possible someday to build a skyscraper even taller than a mile-high building? Probably yes. Since

[3] Al-Kodmany, K., 2011, "Tall buildings, design, and technology: visions for the twenty-first century city," Journal of Urban Technology, Vol. 18, No. 3, pp. 115–140.

1885 when the 10-storey Home Insurance Building was built in Chicago, we have come a long way to the 2,717ft (828m) tall Burj Khalifa in Dubai, 2010. Will building height increase at this rate in the future? We may never know and may not contemplate this at present but we cannot guarantee that this will not definitely happen. One thing we know is that an inborn human instinct is to overcome obstacles through creativity and dreaming. This raises fundamental philosophical and moral questions about the so-called cloud-piercing future visionary towers.

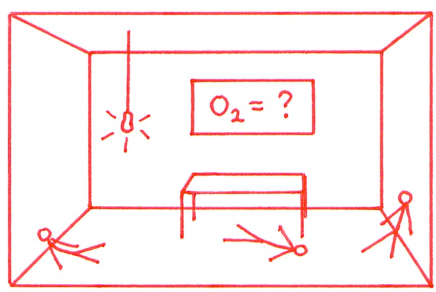
So the megastructures of the future may yet dwarf the tallest buildings of today. People have different views about ultra-tall buildings, but I find them remarkable and fascinating. I look forward to the mile-high buildings of the future.

14.2 How long do we have left? ★★

A good friend of mine used to test prospective chemistry students with this estimation question back in the 1970s and 1980s. In those days he was a professor of physical chemistry and was researching the respirative absorption of oxygen and other gases, as applied to anaesthetics and diving. He must have been doing something right because one of his students went on to win the Nobel Prize for chemistry.

The question relies on a basic knowledge of physiology, chemistry and physics, at lower than pre-university standard. Anyone interested in the world around us should be able to make at least a good *guesstimate*[4] of most of the quantities required. The question went as follows.

THERE ARE THREE of us in this room (whatever room you happen to be in). Assuming it to be perfectly sealed, how long will we live for?



[4]Better than a guess, but worse than an estimate.

The Deadly Game of Puzzle Points

"Rébuffat points" must have been responsible for the deaths of quite a number of young mountaineers. In his 1973 book The Mont Blanc Massif: The Hundred Finest Routes, Gaston Rébuffat—French Alpinist extraordinaire—made the fatal error of organising his climbs roughly in order of difficulty, from route one to one hundred. Mountaineers are no less competitive creatures than mathematicians, and the deadly game of collecting Rébuffat points was quickly established. You got one point for a gentle traverse of the Clocher Clochetons on the Aiguilles Rouges, the first route in his book, and 100 points for dicing with death on the unrelentingly vertical walls of the Central Pillar of Freney, the last route in the book. If you got back alive, you could casually drop into conversation that you had "had a good season". If you were lucky, someone would ask how many Rébuffat points you had collected, and you could brag that you had collected x points in that particular year, giving you y points in total. It was difficult to know whether to go for broke on a few hard routes or tick off many easy ones.

So it is with Puzzle Points.

"The four-star puzzles are just so much harder than the one-star puzzles," I said to my friend Tet. "It would seem unfair to give just four times as many points."

"Then square the number of stars before adding," said Tet.[1]

And so the Deadly Game of Puzzle Point collecting was born.

The distribution of questions in this book (including optional questions for which solutions are given) is: 35 ★, 40 ★★, 28 ★★★, and 10 ★★★★. That makes 113 questions in total, and a potential 607 Puzzle Points if we sum the squares of the number of stars.[2]

"What's the point of points if you can't brag about them?" said Tet.

And so the Puzzle Hall of Fame was born. By keeping track of your

[1] In fact, a similar method of scoring was used for many years in Oxford maths exams. The raw marks for each question were squared before being added. Like any other arbitrary method of marking exams, it favours a certain kind of approach to the exam. In this case, it favours deep knowledge of particular topics rather than superficial knowledge of many topics.

[2] You calculate your points like this: $(35 \times 1^2) + (40 \times 2^2) + (28 \times 3^2) + (10 \times 4^2) = 607$.

progress with the ticklists and scoresheet on the pages that follow, you can calculate your Puzzle Points and your honorary title.

- < 35 Puzzle Points: Aspirant Puzzlist
- $35 \leq$ Puzzle Points < 195: Novice Puzzlist
- $195 \leq$ Puzzle Points < 447: Expert Puzzlist
- $447 \leq$ Puzzle Points < 607: Puzzle Master
- 607 Puzzle Points: Puzzle Overlord

You can enter and update your points in the Hall of Fame by going to the website created for this book:

www.PerplexingProblems.com

Here you can debate the questions, make the case for alternative or more elegant solutions, and propose questions of your own. You might think it strange, but I have avoided "clever" solutions in favour of more standard methods throughout this book. Of course, I appreciate clever solutions as much as the next person, but I thought it would be more instructive to focus on the basic techniques, especially as we're dealing with such unusual questions. There is ample scope, therefore, to "improve" upon many of my solutions in terms of simplicity and compactness. I look forward to seeing these "clever" solutions on the website.

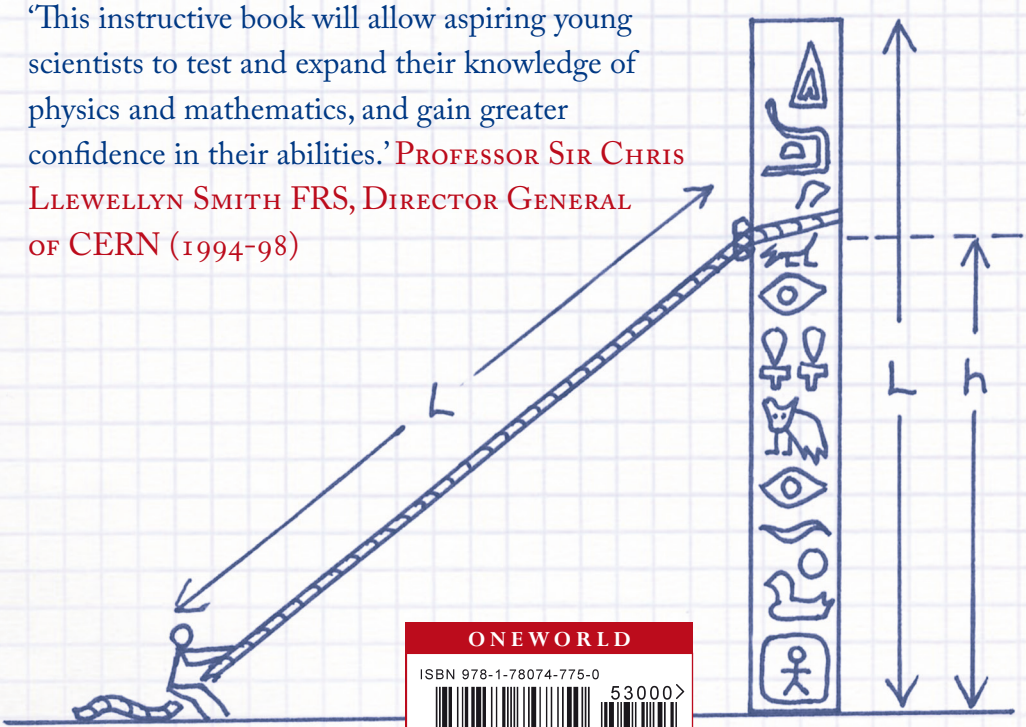
Like all adventures, these puzzles are much more fun when they are shared. Hopefully you already have someone to share your puzzling with, but the Puzzle Forum on the website allows you to share your ideas with an even wider audience, and—more importantly—to learn from others. From time to time I will also post new questions on the website.

Finally, good luck. And I hope you enjoy the adventure, wherever it takes you.

'Entry to selective universities will often require students to demonstrate that they have engaged in super-curricular activities which develop their awareness and understanding of their subject. Tom Povey has provided in this book enough super-curricular content to keep an aspiring mathematician, physicist, engineer or material scientist (and their teachers) happy for months. For a parent or teacher who wants to pro-actively support a student in preparing for entry to a competitive science or maths course this is a no-brainer purchase.' Mike Nicholson, Director of Undergraduate Admissions and Outreach, University of Oxford (2006-14), Director of Student Recruitment and Admissions, University of Bath

'This book is about the excitement of mathematics and physics at the school/university boundary. It contains a remarkable collection of intriguing intellectual challenges, all with complete solutions. You can read it to increase your ability in maths and physics; you can read it to develop your problem-solving skills; but most of all you can read it for fun.' Professor Richard Prager, Head of the School of Technology, University of Cambridge

'This instructive book will allow aspiring young scientists to test and expand their knowledge of physics and mathematics, and gain greater confidence in their abilities.' Professor Sir Chris Llewellyn Smith FRS, Director General of CERN (1994-98)



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